PHILOSOPHICAL

TRANSACTIONS,

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCXIX.

PART II.

LONDON,

PRINTED BY W. BULMER AND CO. CLEVELAND ROW, ST. JAMES'S;
AND SOLD BY G. AND W. NICOL, PALL-MALL, BOOKSELLERS TO HIS MAJESTY,
AND PRINTERS TO THE ROYAL SOCIETY.

MDCCCXIX.

PHILOSOPHICAL

TRANSACTIONS

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCXIII.

PART II.

LONDON.

CONTENTS.

XII. *On the specific gravity, and temperature of Sea Waters, in different parts of the Ocean, and in particular seas; with some account of their saline contents.* By Alexander Marcet, *M. D. F. R. S. &c.* p. 161

XIII. *An account of the fossil skeleton of the Proteo-saurus.* By *Sir* Everard Home, *Bart. V. P. R. S.* p. 209

XIV. *Reasons for giving the name Proteo-saurus to the fossil skeleton which has been described.* By *Sir* Everard Home, *Bart. V. P. R. S.* 212

XV. *Some Observations on the peculiarity of the Tides between Fairleigh and the North Foreland; with an explanation of the supposed meeting of the Tides near Dungeness.* By James Anderson, *Captain in the Royal Navy. Communicated by the Right Hon. Sir* Joseph Banks, *Bart. G. C. B. P. R. S.* p. 217

XVI. *On the Ova of the different tribes of Opossum and Ornithorhynchus.* By *Sir* Everard Home, *Bart. V. P. R. S.* p. 234

XVII. *The results of Observations made at the Observatory of Trinity College, Dublin, for determining the Obliquity of the Ecliptic, and the Maximum of the Aberration of Light.* By the *Rev.* J. Brinkley, *D. D. F. R. S. and M. R. I. A. and Andrew's Professor of Astronomy in the University of Dublin.* p. 241

XVIII. *On some new Methods of investigating the Sums of several Classes of infinite Series.* By Charles Babbage, *Esq. A. M. F. R. S.* p. 249

XIX. *On the optical and physical properties of Tabasheer. By* David Brewster, *LL. D. F. R. S. Lond. and Edin. In a Letter to the Right Hon. Sir* Joseph Banks, *Bart. G. C. B. P. R. S. &c. &c. &c.* p. 283

XX. *An Account of a Membrane in the Eye, now first described. By* Arthur Jacob, *M. D. Member of the Royal College of Surgeons, Ireland; Demonstrator of Anatomy, and Lecturer on the Diseases of the Eye in the University of Dublin. Communicated by* James Macartney, *M. D. F. R. S.* p. 300

XXI. *A new method of solving numerical equations of all orders, by continuous approximation. By* W. G. Horner, *Esq. Communicated by* Davies Gilbert, *Esq. F. R. S.* p. 308

PHILOSOPHICAL TRANSACTIONS.

XII. *On the specific gravity, and temperature of Sea Waters, in different parts of the Ocean, and in particular seas; with some account of their saline contents.* By Alexander Marcet, *M. D. F. R. S. &c.*

Read May 20, 1819.

WHILE analyzing the waters of the Dead Sea and the River Jordan, about twelve years ago, and conversing on the peculiarities of these waters with a late valuable and lamented Member of this Society, Mr. TENNANT, it occurred to us that a chemical examination of different seas, in a variety of latitudes and at different depths, might be interesting; and that, however unlikely to be productive of any striking discovery, such an inquiry, conducted with due care and attention, might afford curious results, and throw some light on this obscure subject. We accordingly began to collect specimens of sea water from various parts of the globe, and it was agreed that I should, aided by Mr. TENNANT's occasional advice, submit them to chemical anaysis.

In the course of a few years I became possessed, through

the kindness of several friends, of a great variety of specimens of sea water; and I was preparing to examine them, when a most deplorable accident deprived science of the sagacious philosopher from whose friendship and enlightened assistance I had anticipated so much advantage. Procrastination and delay were the natural consequence of this misfortune; and I should probably have entirely lost sight of the subject, had not my intention been again directed to it by the late expeditions to the Arctic regions, and the great zeal and kindness of some of the officers engaged in them, in procuring for me specimens of sea water, collected in different latitudes, and under peculiar circumstances, so as to add greatly to the value of those which I previously possessed.*

I must not omit to observe, however, that this subject has, at various periods, engaged the attention of philosophers. Thus Bergman, † Watson, ‡ Nairn, § Bladh, || Lavoisier, ¶ Pagès,** Phipps,†† Litchtemberg,‡‡ Pfaff,§§ Bouillon-La Grange and Vogel,||| &c., have turned their attention to the subject of sea water, and ascertained some valuable, though in general detached and often discordant facts; and more lately the celebrated traveller Humboldt,¶¶ Dr. Murray of

* I am also indebted, both to Sir Joseph Banks and to the British Museum, for various specimens of water from the same expedition.

† Bergman's Opusc. Vol. I. ‡ Watson's Chemical Essays, Vol. V. p. 92.

§ Philos. Trans. for 1776. || Kirwan's Geological Essays, p. 350.

¶ Lavoisier's Memoirs, 1778.

** Pagès' Voyage round the World, from 1767 to 1771.

†† Phipps's Voyage towards the North Pole, 1773.

‡‡ Litchtemberg, in Schewigger's Journal II. 256. §§ Pfaff, ibid.

||| Annales de Chimie, Vol. LXXXVII. ; and Ann. of Philos. IV. 200.

¶¶ Humboldt, Relat. Historique, Vol. I.

Edinburgh,* Dr. JOHN DAVY, now of Ceylon,† and M. GAY-LUSSAC,‡ have also added many useful facts to this part of natural science. These two last observers, in particular, have given the specific gravities of waters in a variety of latitudes, from England to Ceylon, and from South America to France, and their results appear to lead to the general conclusion, that the variations obtained in those parts of the ocean were but very slight, and are to be ascribed rather to accidental causes than to any general principle.§

The immediate objects of investigation which presented themselves to me were, first, to ascertain the specific gravities of upwards of seventy specimens of sea water which I had procured from different parts of the world; and afterwards to examine whether any differences could be traced in the chemical composition of those waters. This naturally gave rise to two distinct parts, and afforded a convenient mode of dividing the subject.

§ 1. *Of the specific gravities of sea waters, from different seas, and in different latitudes; with some remarks on the temperature of those seas.*

Before I proceed to state the results, I shall briefly describe the mode in which the waters were weighed, and the apparatus which I contrived for the purpose of raising water from the bottom of the sea.

The specific gravities were taken in the usual mode, that is, by comparison with equal bulks of distilled water of the same

* Edinb. Philos. Trans. Vol. VIII. † Philos. Trans. for 1817.

‡ Ann. de Chim. for Dec. 1817; and Philos. Magaz. Vol. LI.

§ M. De HUMBOLDT (*Personal Narrative, Vol. I.*) thought he could perceive that

temperature.* The balance I used was one which was sensibly affected by $\frac{1}{10}$th part of a grain; but I did not think it necessary to use smaller weights than $\frac{1}{20}$th part of a grain: so that whenever, in the annexed tables, smaller weights are expressed, in the sixth decimal figure, such very minute parts must not be understood to have been obtained from actual experiment, since they resulted, by calculation, from the conversion of the weights actually obtained into the usual standard of 1000 parts.

The first idea of the apparatus which I contrived for raising water from the bottom of the sea, occurred to me about ten years ago, on accidentally seeing in an instrument maker's shop, a machine, said to be the identical one which was used for a similar purpose by Dr. Irving, in Captain Phipps's (since Lord Mulgrave) expedition.† This consisted simply in a cylindrical vessel having an opening at the top, and a similar one at the bottom, each closed by a flap or valve opening only upwards, and moving freely upon hinges. When this

sea water was less salt between the tropics than between the coasts of Spain and Teneriffe; and his observations seem to lead to the inference, that there is a specific gravity peculiar to the water of each zone; a conjecture, however, which the facts collected in this paper do not appear to confirm.

* It may perhaps be worth while to mention a small improvement which was introduced in the vessel used for weighing the waters. The apparatus consisted in a thin phial, nearly spherical, containing between five and six hundred grains of distilled water, and having a very light ground glass stopper. But as I had observed on former occasions that such phials were apt to burst on the stopper being forced in, from the compression of the liquid with which they were filled, I had the stopper made with a very small longitudinal aperture through it, so as to allow a minute quantity of water to ooze out; and this was very easily accomplished (by Mr. Newman, of Lisle-street) by forming the stopper of a portion of thermometer tube, the bore of which perfectly answered the desired purpose.

† Phipps's Voyage to the North Pole in 1773.

apparatus was sunk into the sea, the valves would, of course, be kept open by the current of water passing freely through the machine so long as it descended; and when drawn up again, the valves would be kept closed by the water acting in an opposite direction. As however, in heaving a machine of this kind from a great depth, it is almost impossible that some oscillations should not take place in it, either from the motion of the boat, or from some accidental jerks in pulling the line, it is evident that these oscillations being necessarily communicated to the valves, partial changes of the water contained in the vessel are almost unavoidable. It was principally with a view to remove this objection, that I modified the principle of the apparatus in the following manner. I procured a strong cylindrical brass vessel, capable of holding about half a pint of water, and having, like Dr. IRVING's machine, an aperture both at the top and bottom, with a valve at each aperture opening upwards. But these valves, instead of being allowed to move backwards and forwards freely upon their hinges, were, when closed, firmly locked by springs; and when open, could only be kept in that state by the application of a certain degree of force. This force consisted in a weight of several pounds, suspended to the extremity of a cord, the other end of which was fastened to the valves; and the machine was so constructed (as will be readily understood by referring to the annexed sketch, Plate XI.), that the moment the valves were relieved of the weight which kept them open, they closed instantly, and were forcibly locked by the above mentioned contrivance. Now the machine being open, with the weight suspended to it at a few feet distance, and the whole being let down into the sea, it is evident that the apparatus must continue perfectly open

and pervious till the weight touches the bottom, at which moment it closes itself, and remains, in that state. The water brought up in this machine, therefore, can only come from the bottom, and from no intermediate depth; for whenever the apparatus fails in reaching the bottom, it continues pervious, and on being raised out of the water is found open, and emptied of its contents. The advantage however of raising water from the bottom with certainty, whenever the bottom can be reached, is a valuable one, and has already enabled me to ascertain some curious points respecting the sea of Marmora, as will be presently seen; and if some mode could be contrived of closing the machine at any desired intermediate distance from the surface (an object to which I have directed the attention of the ingenious instrument-maker, Mr. NEWMAN), it it would then answer every desired purpose.*

* Since this paper was written, Mr. NEWMAN has succeeded in constructing a machine, (just in time to be put on board of Lieut. PARRY's ship, about to sail to Baffin's Bay), which seems well calculated to answer the desired object, under any circumstances. The principle is essentially the same as that of the machine which I have just described, namely, that of closing itself when it touches the bottom; but with this material addition, that the valves, when the bottom cannot be reached, may be closed at any given depth, by causing a weight to descend along the cord to which the machine is suspended, till it comes into contact with it, and closes the valves by an appropriate contrivance. A sketch of this machine, with its explanation, is also annexed at the end of this paper, Plate XII.

Among the attempts which have been made to bring up water from any given depth, I should not omit to mention an ingenious contrivance of Mr. TENNANT, which he thought calculated to answer the purpose tolerably well, and which, as will be seen hereafter, was used some years ago with apparent success.

Mr. TENNANT's contrivance consisted in a wooden box, capable of holding a few ounces of water, and opening like a snuff-box, by a lid moving upon hinges, and fitting water-tight. The box, when closed, was forcibly kept in that state by a spring; but when about to be used, it was partially opened, and was prevented from shutting itself

I shall now proceed to state the results which I have obtained in regard to the specific gravities of sea water; and for the sake of conciseness I shall present them in the form of

again by means of a small spring or wedge. It remained open so long as it descended; but when pulled up again, the wedge was thrown out of its place, and this obstacle being removed, the box closed itself instantly. This was effected by means of a small fly-wheel, so confined as to admit of no motion during the descent of the machine; but the moment it began to ascend, the pressure of the water communicated to the wheel a rotatory motion, which, through a little mechanical contrivance, disengaged the spring and closed the box.

Another, and more elaborate invention, of the same kind (a description of which may be found in the 'Journal of Science and the Arts, Vol. V.) was devised last year by Sir HUMPHRY DAVY, and was repeatedly used in the late voyages towards the North Pole; and in fact, many of the specimens of waters from those regions, which I have examined, were raised by means of that ingenious, though I apprehend, not unexceptionable apparatus.

The principle of this instrument may be stated in a few words. It consists in a strong copper bottle of an oblong shape, closed at its neck by a stop-cock. To this bottle is attached laterally, and in a parallel direction, a metallic tube closed at the top and open at the bottom, with an air-tight piston moving within the tube. As the open end of the tube therefore descends into the sea along with the bottle, the piston which closes the orifice of the tube is gradually forced upwards into it, as the machine sinks, the air within it being proportionally compressed; but when the piston has reached a certain part of the tube, it meets with a catch and opens the cock of the bottle, which, of course, instantly fills with water; and there is an ingenious contrivance by which the machine may be set before hand, so as not to let in the water till a certain known degree of pressure is made by the superincumbent column.

Captain Ross, in his account of his 'Voyage to the Arctic regions' (Appendix, page cxxiv.), makes the following observation: 'Sir HUMPHRY DAVY's apparatus answered the purpose for which it was intended; but it did not close so as to prevent the water from escaping or mixing with that nearer the surface as it came up.' This objection however, which might probably be removed, is of no great importance, as the aperture in question being very small, and situated at the upper end of the vessel, there is no risk of any sensible change taking place in the contents of the vessel through that aperture, while traversing the upper strata. But the machine appears liabl to the

tables, taking care to annex to each specific gravity, under the head 'observations,' any collateral information which I may be able to offer.

more serious objection of depending for accuracy upon very nice adjustments, which can hardly be relied upon under very great pressures.

Lieut. Franklin, who commanded his Majesty's ship the Trent, and was not provided with one of Sir Humphry Davy's machines, sometimes used a cylindrical leaden vessel with two valves, like that employed by Dr. Irving; and at other times he made use of an empty corked bottle, the cork being strongly tied to the bottle. In every instance the bottle was found filled, the cork having been forced into the bottle; except on one occasion, on which the bottle had filled itself without the cork being driven in, not apparently from its being more strongly fastened than in other cases, but rather probably from its being more porous, so as to allow the water to filter through it before the pressure was sufficient to move it from its position. Lieut. Franklin had with him my instrument; but he used it only in a few instances, on account of its being too light to reach great depths, and of too delicate a structure to render the addition of a very large weight practicable; an objection however which may very easily be remedied. From all these circumstances it is easy to perceive, that the means used for raising water from great depths, have hitherto been far from uniform in their principle, or certain in their performance.

TABLE I. *Specific Gravities of Sea Waters.*

Designation of Seas.	Nos. of Specimens.	Latitude.	Longitude.	Specif. Grav.	OBSERVATIONS.
Arctic Ocean.	1	66,50	68,30W	1025,55	Taken up by Captain Ross, in Sept. 1818, from a depth of 90 fathoms, with Sir Humphry Davy's apparatus. Temperature of the water at 90 fms. 30°; at 200 fms. 29°; at 400 fms. 28,5°; at 670 fms. 25°; at the surface 33°. Temperature of the air, 36°. Bottle labelled in Capt. Ross's own hand-writing, with all the above particulars.
	2	74,0	—	1025,46*	By Lieut. Parry, from the surface. The ship surrounded by ice in every direction. Temperature of the water 31°, of the air 34°; 8 July, 1818.
	3	74,50	59,30	1026,19	By Lieut. Parry. Temperature of water 39°, of air 38°.
	4	75,14	4,49E	1027,27	By Lieut. Franklin, from the surface, 10 Sept. 1818.
	5	75,14	4,49	1027,27	By Lieut. Franklin, raised with the cylindrical machine, from a depth of 736 fms. Temperature of the water brought up 36°, of the air 35°; 10 September, 1818.
	6	75,54	65,32W	1022,7 *	By Captain Ross, from the surface, 4 miles from the land; 12 August, 1818.
	7	75,54	65,32	1025,9	By Capt. Ross; from a depth of 90 fms. with Sir H. Davy's machine. Soundings 138 fms. 12 August, 1818.
	8	76,32	76,46	1024,05*	By Capt. Ross; from the surface. Soundings 100 fms. 22 August 1818.
	9	76,32	76,46	1026,22	By Capt. Ross; from a depth of 90 fms. Temperature 30,5°; 22 August, 1818.
	10	76,33	—	1026,64	By Lieut. Parry; with Sir H. Davy's machine, from a depth of 80 fms. Temperature of the water 29°, of air 30°; 21 Aug. 1818.
	11	79,37	11,15E	1026,7	By Lieut. Franklin, from a depth of 34 fms. Temperature of the sea at the surface 30,2°, at 34 fms. 33,2°; of the air 33,2°.
	12	80,26	10,30	1022,55*	By Lieut. Franklin, 13 July, from the surface; ship beset with ice; 12 leagues from the Coast of Spitzberg. Temperature of the surface, 32,5°, of air 36°.
	13	80,26	10,30	1027,14	By Lieut. Franklin; from the bottom, depth of 287 fms.
	14	80,26	10,30	1027,15	By Lieut. Franklin: from the bottom, depth of 287 fms. with Dr. Marcet's machine. Temperature of the bottom 33,5°; 13 July, 1818.
	15	80,28	10,20	1026,8	By Lieut. Franklin; from bottom, depth of 185 fms. surface being frozen. Temperature of the bottom 30½°, surface 32½°; 13 July, 1818.
	16	80,29	11,0	1026,84	By Lieut. Franklin; from a depth of 303 fms. being the bottom. Temperature of the air 35°, of the surface of the sea 32,9°; 18 July, 1818.

N. B. The specimens marked * in the three first tables, cannot be taken into account in calculating the mean specific gravity of the waters of the ocean, their saline contents being much diminished either by the vicinity of large masses of ice, or of great rivers, which reduce them much below the average standard of density of sea water.

TABLE II. *Specific Gravities of Sea Waters.*

Designation of Seas.	Nos. of Specimens.	Latitude.	Longitude.	Specific Gravity.	OBSERVATIONS.
Northern Hemisphere.	17	63,49	55,38W	1026,7	By Lieut. Parry, in July, 1818; from a depth of 80 fms. Temperature of the water 32½° of the surface 33°, of the air 29½°
	18	59,40	14,46	1030,04	By Capt. Basil Hall; from the surface. In July, 1811.
	19	56,22	—	1026,56	Taken up by Dr. Berger, about 15 leagues from the West Coast of Jutland; depth 28 fms. Dec. 1810.
	20	54,0	4,30	1026,8	By Dr. Berger; Calf of Man, Irish Sea.
	21	53,45	0,20	1026,7	By Dr. Berger; near Hull.
	22	52,45	4,0	1021,75*	By myself; from Barmouth, Wales, near the mouth of the river Mawdack.
	23	48,25	6,34	1030,02	From Mr. Tennant; taken up by Mr. Lushington.
	24	46,0	48,0	1026,48	By Mr. Caldewell, Coast of Canada. Temperature of water 42°, of air 50°.
	25	45,2 0	45,10	1028,16	By Mr. Caldwell; brought up from a depth of 230 fms. by means of a corked bottle.
	26	45,10	15,0	1029,34	By Capt. Hall, in January, 1811.
	27	25,30	32,30	1028,86	By Capt. Hall; nearly in the middle of the North Atlantic.
	28	22,0	89,0E	1020,28*	By Capt. Hall; from the mouth of the Ganges, about 90 miles from Calcutta. Water muddy.
	29	13,0	74,0	1027,72	By Capt. Hall; Coast of Malabar, off Cochin; some sediment, apparently vegetable.
	30	10,50	24,26W	1028,25	By Mr. Schmidtmeyer, going to South America; bottle blackened, smell hepatic.
	31	7,0	80,E	1030,9	From Mr. Tennant, by Mr. Lushington, off Colombo, Ceylon.
	32	4,0	23,W	1027,72	By Mr. Schmidtmeyer, in April, 1808. Therm. 84°.
	33	3,28	81,4E	1030,22	From Mr. Tennant, by Mr. Lushington.
Equator.	34	0,	25,30W	1028,25	By Mr. Schmidtmeyer.
	35	0,	23,0W	1027,85	By Capt. Hall, in August, 1817.
	36	0,	83,0E	1028,07	By Capt. Hall, in 1813; about 300 miles south of Ceylon.
	37	0.	92,0E	1026,92	By Capt. Hall; 2 or 400 miles west of Sumatra, June, 1817.

The specific gravity of the specimens marked * in this and the following Table, being obviously much less than common, in consequence of the vicinity of rivers, these specimens have not been taken into account in calculating the mean specific gravity of sea-water.

TABLE III. *Specific Gravities of Sea Waters.*

Designation of Seas.	Nos. of Specimens.	Latitude.	Longitude.	Specific Gravity.	OBSERVATIONS.
Southern Hemisphere.	38	8,30 S.	32,0 W.	1028,95	Taken up by Mr. Schmidtmeyer, in May 1808. Temperature 83°.
	39	9,0	35,0	1029,20	By Mr. Schmidtmeyer, at Pernambocco.
	40	11,30	33,7	1029,80	From Mr. Tennant, by Mr. Lushington.
	41	21,0	0,	1028,19	By Capt. Hall, near the middle of the South Atlantic.
	42	23,30	73,0 E.	1028,31	By Capt. Hall, Tropic of Capricorn, between Madagascar and New Holland.
	43	25,30	5,30	1032,09	By Capt. Hall; about half way between St. Helena and the Cape; in June, 1815.
	44	28,0	43,0	1027,15	By Capt. Hall; Mosambique, South of Madagascar.
	45	35,0†	56,0 W.	1025,45	From Mr. Tennant, by Mr. Lushington; mouth of the Rio de la Plata.
	46	35,10	21,0E.	1027,5	By Capt. Hall; South of the Cape, on the Banks of Lagullas.
	47	35,33	0,21	1031,6	From Mr. Tennant, by Mr. Lushington; phial partly emptied.
Yellow Sea.	48	35,0 N.†	—	1022,91	By Captain Hall, in 1816. There were several phials of this water, with glass stoppers. All the phials were blackened internally by the water, which had a highly hepatic smell. This water, when seen in large masses, has a greenish yellow colour.
Mediterranean.	49	36,0N.†	5,0W.	1030,1	By Dr. Macmichael, in 1811, from a depth of 250 fms. in the Straits of Gibraltar, between Cape Europa and Cabrita, with Mr. Tennant's machine.
	50	36,0N.†	5,0	1030,5	By Dr. Macmichael; from the same spot as the preceding, but from the surface.
	51	—	—	1027,3	By Mr. Tennant; taken up by himself at Marseilles, in 1815; Latitude not specified.

† The Latitudes thus marked are stated only as approximations, not being specified on the labels of the bottles.

TABLE IV. *Specific Gravities of Sea Waters.*

Denomination of Seas.	Num. of Specimens.	Latitude.	Longitude.	Specific Gravity.	OBSERVATIONS.
Sea of Marmora.	52	40,5 N	26,12 E	1028,19	Taken up by Sir Robert Liston, at the entrance of the Hellespont or Dardanelles, *from the bottom*, 84 fms. deep, by my machine, in June 1813.
	53	40,5	26,12	1020,28	By the same Gentleman, and exactly from the same spot as the preceding, but *from the surface*.
	54	41,0†	29,0	1014,44	By Sir Robert Liston; at the entrance of the Bosphorus or North entrance of the channel of Constantinople, about four miles from the land; *from the bottom*, 20 fms.
	55	41,0†	29,0	1013,28	By the same Gentleman; same spot, but *from the surface*.
Black Sea.	56	—	—	1014,22	By Mr. Sautter; one of the specimens clear; the other slightly hepatic. Latitudes not stated.
	57	—	—	1014,14	
White Sea.	58	65,15 N	39,19E	1018,94	By Mr. Sautter, in 1811. Water perfectly clear.
	59	—	—	1019,09	By the same; latitude not noted.
Baltic.	60	56,0N	15,0 E	1004,9	By Mr. Prevost, in Carlsham harbour; cork and bottle slightly blackened.
	61	57,39	—	1025,93	By Dr. Berger; in 1810, Categat, one mile and a half from the Eastern coast of Jutland. Depth about 14 fms.
	62	56,0	12,40E	1015,87	By Dr. Berger; from the Sound, or Passage into the Baltic, half-way between Denmark and Sweden. Depth about 17 fms.
Ice-Sea Waters.	63	75,54N	65,32 W	1000,	By Captain Ross; from the same spot as No. 6 and 7; sounding 250 fms. from an Iceberg. 19th August, 1818.
	64	80,28	10,20E	1000,17	By Lieutenant Franklin, from *water at the surface*, when beset amongst ice. Same spot as No. 15. Temper. of the surface 32. 8°.—15 July, 1808.
	65	79,56	11,30	1000,6	By Lieutenant Franklin, from a floe, the ice being 14 feet deep under the surface; 21st. of June, 1818.
	66	79,38	11,0	1000,15	By Lieutenant Franklin, from an immense iceberg. August, 1818.
	67	76,48	13,40	1002,35	By Lieutenant Franklin, on the 28th May, 1818. About 20 miles from Spitzberg, Temp. of air 29°. Soundings 600 fms. Taken from the surface of a small detached piece of ice floating in the sea.
	68	75,40	61,20 W	1000,15	By Lieutenant Parry, from young ice on the surface, about ¾ inch thick, July 21, 1819.

† The Latitudes and Longitudes thus marked were inferred from the description of the spot, not being stated on the bottles.

In endeavouring to connect together the various statements contained in the above tables, the following inferences present themselves.

The ocean in the southern hemisphere, would appear to contain more salt than in the northern hemisphere, in the proportion of 1029,19 to 1027,57; as may be seen by taking the mean specific gravity of the waters collected from the two hemispheres. But it must be observed, that a great proportion of the specimens from the northern hemisphere were taken farther from the equator than those procured from the other hemisphere, which may possibly account for the difference in question.*

The mean specific gravity of specimens taken from various parts of the equator is 1027,77, and is therefore a little greater than that which prevails in the northern hemisphere, though sensibly less than that of the southern ocean.

There is no notable difference between different east and west longitudes at the equator; nor is there, in other latitudes, any material and constant difference between waters of the ocean in corresponding east or west longitudes in the same hemisphere.

There is no satisfactory evidence of the sea, at great depths, being more strongly impregnated with salt than it is near the surface; except under peculiar circumstances, which will

* It may be observed also that Dr. Davy, in the experiments abovementioned (Philos. Trans. 1817), generally found the specific gravity of sea water, both in the South Atlantic and in the Indian Ocean, lower than I have done; for which I am at a loss to assign any reason, unless it be supposed that some of my specimens, from having been long kept, and perhaps not corked with sufficient care, may have undergone some degree of concentration.

hereafter be explained, and appear independent of any general law.

In general the waters of the ocean, whether taken from the bottom or from the surface, appear to contain most salt in places in which the sea is deepest or most remote from land; and the vicinity of large masses of ice seems to have a similar effect to that of land in diminishing the saltness of the sea. If, therefore, in attempting to approach the Pole, the saltness of the sea should appear to increase, and become more uniform at the surface, such a circumstance might be considered as militating against the probability of the sea being extensively frozen in those regions.

It may be stated generally, that small inland seas, though communicating with the ocean, are much less salt than the open ocean. This is particularly striking in the case of the Baltic; and also, though in a less remarkable degree, in the Black sea, in the White sea, in the sea of Marmora, and even in the Yellow sea.*

The Mediterranean, though a comparatively small and subordinate sea, is found to contain rather a larger proportion of salt than the ocean.† This appears to form an exception

* The Caspian sea is also said, but upon no certain authority, to be less salt than the ocean. Its waters having, like those of the Dead sea, no obvious communication with those of any other seas, present a particular case well deserving of investigation; and I regret that I have not yet been able to procure a specimen of them, notwithstanding the various attempts which I have made for that purpose.

† This has been stated by various writers, and appears to be the case from the few specimens which I have examined; but I cannot speak with perfect confidence on this point, as I was but scantily supplied with water from that sea, though comparatively so near and so much frequented by navigators of all descriptions. In their analysis of sea water, Bouillon-La Grange, and Vogel state the proportion of saline matter in the water of the Mediterranean to be 41, that of the Atlantic being 38, and that of the English Channel 36.

to the general fact which I have stated above, and stands in need of explanation.

In order to account for this, it has been argued that the Mediterranean sea is not supplied by the rivers which flow into it with a quantity of fresh water, sufficient to replace that which it loses by evaporation under a burning sun, assisted by a powerful radiation from the African shores, and the parching winds blowing from the adjacent deserts; so that a current from the ocean is required to replenish this waste, and prevent the Mediterranean from sinking below its level. Accordingly it is observed that a current sets in at all times from the ocean into the Mediterranean, which current, I am informed, is so strong at Gibraltar, as to carry a ship at the rate of two or three miles an hour; and it is felt as far eastward as the Cap de Gat, a distance of upwards of one hundred and fifty miles; so that ships going out of the Mediterranean, scarcely ever attempt to beat out against contrary winds, and usually keep close either to the African or European shore, in order to avoid the full force of the stream.

If this hypothesis, however, of a disproportion between the loss of water produced by evaporation, and the inadequate compensation afforded by the ingress of rivers, be founded in fact; and if this deficiency be replenished by the saline waters of the ocean, it will be necessary to explain why the waters of the Mediterranean do not gradually increase in saltness, and indeed how it happens that they are not ultimately converted into a saturated brine. It has been supposed, in order to remove this difficulty, that an under-current of water, salter than the ocean, runs out of the Mediterranean at the Straits

of Gibraltar, and unloads its waters of their excess of salt. But however plausible this theory may be, it must be confessed, that scarcely any other argument has hitherto been alledged in support of the probability of this under-current, than the easy explanation it would afford of the phenomena, and analogies derived from the familiar fact of opposite atmospheric currents formed in confined places, from the mere admission of air of a different temperature.* The following fact, however, for which I am indebted to Dr. MACMICHAEL, who had it from very respectable authority (the British Consul at Valentia), seems to give considerable support to the above theory. Some years ago a vessel was lost at Ceuta, on the Coast of Africa, and its wreck afterwards thrown up at Tariffa, on the European shore, full two miles west of Ceuta. How can this be explained, except by the action of what may be called a counter-submarine current, which would carry a body, sunk to a considerable depth, out of the Straits?

It was a favourite scheme of the late Mr. TENNANT, to examine specimens of sea water from the Straits of Gibraltar, taken both from the surface and from some great depth, in order to ascertain whether the latter would have a greater specific density than the former, a circumstance which, if it did not establish the truth of the theory in question, would at least render it very probable. It was with a view to decide this point, that Mr. TENNANT constructed the machine which

* Thus it is well known that if the door of a heated apartment be partially opened, and two lighted candles placed the one at the top and the other at the lower part of the aperture, the uppermost candle will have its flame propelled outwards, by the rushing out of the heated and therefore lighter current, while the other candle will have its flame blown inwards by the opposite effect.

I have before mentioned, by means of which he flattered himself that water could be brought up from any desired depth; and it was upon the same occasion that I contrived the apparatus above described, in hopes that it would enable me to obtain water from the bottom of the Straits. My friend, Dr. MACMICHAEL, one of the travelling Fellows of the University of Oxford, and Member of this Society, undertook to make the attempt. He succeeded in procuring water in the Straits, from the depth of two hundred and fifty fathoms, with Mr. TENNANT's machine; but all attempts to obtain water from the bottom proved fruitless, from the impossibility of reaching it on account of the very great depth of the sea in that spot.* The specimens of water, however, procured by Mr. TENNANT's machine, were sent home, and were soon afterwards examined, in the presence of Dr. MACMICHAEL, by Mr. TENNANT, who could not detect any difference in their specific gravity; and when I lately re-examined the same specimens, which had been preserved, it even appeared (probably from some accidental circumstance) that the specimen from the surface was a little heavier than the other. This point therefore remains to be decided by farther investigation.

With regard to the waters of the Atlantic, although no pains have been spared by the able and zealous officers employed in the late voyages towards the Pole, to procure specimens of water, both from the surface and from great depths, with a view to compare their densities, and though I have

* This attempt was made in Sept. 1811, in the Bay of Gibraltar, between Cape Europa and Cabrita. See Table III.

been favoured with many of those specimens, I have not been able to obtain results sufficiently conclusive to enable me to form a decided opinion upon the subject. On referring to the annexed tables, it will be observed, that, in a variety of instances, the water at the surface was much lighter than when procured from some depth; but then it would appear that whenever such a result was obtained, it was owing to the water at the surface being diluted by the melting of large masses of ice; for under ordinary circumstances (as in the case marked No. 5, Table I.), no such difference was obtained between waters taken at the surface, or brought up from a considerable depth; and in no instance did the density of the water of the Atlantic, from whatever depth it was obtained, appear to exceed the mean density of the waters of the ocean.

The fact however may be, and actually appears to be different in the case of particular seas or arms of the ocean, in which the influence of currents and other local circumstances is more sensibly felt, and the waters of which do not, for obvious reasons, necessarily partake of the uniform saltness of the ocean. The experiment, as was before observed, does not appear to have yet been fairly tried in the Mediterranean, and indeed from the great depth of that sea, it must be extremely difficult, if not impracticable, to raise water from the bottom, at least at any considerable distance from the coast.

In the instance of the sea of Marmora, in which water was obtained with certainty from the bottom, by means of my machine* (Table IV. Specim. 52 to 55), the result was very

* It was through the kindness of the British Ambassador at Constantinople, Sir Robert Liston, that these specimens were procured. Sir Robert has since told me that the use of the machine was not attended with the least difficulty.

remarkable. The difference of specific gravity between the upper and lower strata, at the entrance of the Dardanelles, where the depth is very moderate, proved to be nearly as 1030 is to 1028; a very curious result, which gives additional plausibility to the hypothesis just mentioned respecting the entrance of the Mediterranean.

Among the specimens of sea water collected by Mr. TENNANT, a small phial of water was found, which had been sent him from Persia, by his friend, the unfortunate traveller BROWNE, a short time before he was murdered.* This interesting specimen was taken from the Lake *Ourmia* or *Urumea*,† a small sea situated in the province of Azerbijan in Persia, south-west of Tabreez, and at no great distance from the Volcanic region of Mount Ararat. This lake is thus described by KINNEIR, in his "Geographical Memoirs of the Persian Empire," page 155: "The Lake Urumea, generally believed to be the Spanto of Strabo, and Marcianus of Ptolemy, is 80 fursungs, or according to my computation about 300 miles in circumference. The water is more salt than that of the sea, no fish can live in it, and it emits a disagreeable sulphureous smell. The surface is not however, as has been stated, incrusted with salt; at least it was not so in the month of July, when I saw it; on the contrary, the water was as pellucid as that of the clearest rivulet." Such salt lakes, entirely unconnected with the ocean, being by no means of frequent occurrence, I propose to give, in another part of this paper,

* I was indebted for this water to my friend Mr. WARBURTON, who put it into my hands after Mr. TENNANT's death.

† Called also Lake of Shahee (See MORIER's Second Journey to Persia, page 286).

some account of the saline matter contained in the waters of Lake Ourmia. I shall only state at present that the specific gravity of the specimen of water in my possession, which appears to have been very carefully preserved, is no less than 1165,07, a degree of saline impregnation which has not, I believe, been observed in any other lake, with the exception of the Dead sea, the waters of which are even heavier.

The excellent opportunities which occurred, during the late northern expeditions, for procuring specimens of water from the various kinds of ice which are met with in those regions, and the obliging zeal of its commanders, afforded me the means of making some inquiries into the nature of these waters. With regard to the floating masses of ice called *icebergs*, which are formed from the waters of melted snow, and are detached by rain and torrents, or by their own weight, from the vallies and from precipitous rocks along the shores, it was long known that they consist of fresh water, in a state of great purity, though perhaps seldom so perfectly pure as the specimen marked 63 (Table IV.), the specific gravity of which was exactly 1000. But the immense fields of ice, or *floes*, which are formed from the actual congelation of the surface of the sea, are of a different description. This ice, generally speaking, is not so compact or so transparent as the icebergs, and it is even stated, in a late curious and elaborate dissertation on the subject of the polar seas, published in the Edinburgh Review,* that this ice is

* 'Edinburgh Review', vol. XXX. page 15. There is also in the 4th volume of the 'Journal of Science and the Arts,' a paper of Mr. Scoresby, which was read in 1815, before the Wernerian Society, and contains many curious and valuable observations.

"porous, and consists of spicular shoots or thin flakes, which detain within their interstices the stronger brine; that it can therefore never yield pure water; though if the strong brine be first suffered to drain off slowly, the loose mass which remains will melt into a brackish liquid," &c. This statement however seems to have been founded rather upon results obtained from the artificial congelation of sea water, than from the examination of the sea ice itself; for it will be seen, by a reference to Table IV., that this ice yielded, in every instance, water considerably purer than we commonly find spring water, or even river water.* Thus, for instance, water from young ice scarcely exceeding half an inch in thickness (Table IV. spec. 58), was found to have a specific gravity of only 1000,15; and yet Lieutenant PARRY, by whom this specimen was taken up and brought home, told me that he had not used the precaution of wiping the ice before he suffered it to melt, a circumstance which is more than sufficient to account for the minute quantity of saline matter which it contained.

It appears therefore well established that sea water, when in the act of passing to the state of ice, parts with the whole, or nearly the whole of its salt to the lower and denser strata; and it may be inferred also from several of the results men-

* I found the specific gravity of the water of the Thames, taken from a large cistern in Lombard-street, 1000,43. The water was quite clear; and, from the cistern being filled at different periods of the tide, afforded a good average of that water at London Bridge. I found the specific gravity of the water of the New River, taken from a cistern in my own house, 1000,52; and I was rather surprised to find that the specific gravity of a specimen of spring-water, from a well in Russell square, was only 1000,17.

tioned in the annexed tables, that this separation of saline matter does not exclusively belong to ice actually formed; but that it also prevails, more or less, in water which is only approaching to the state of ice, or has just passed to the liquid state; so that (as appears from specimen 64), there are circumstances under which water may be found on the surface of the sea almost entirely deprived of its saline contents; which fully accounts for the great difference of density observed in the northern ocean, between the surface of the sea and its lower strata. This separation of the saline matter had long been shown, by experiments upon a small scale, to take place during the freezing of sea water; and Mr. NAIRNE, who ascertained this point so far back as the year 1776,* states that this congelation takes place at the temperature of about 28,5°, the water thus frozen being almost entirely freed from its salt. Upon trying a similar experiment with the air pump, in the mode invented by Mr. LESLIE for artificial congelation, I found that I could, without the least difficulty, and in the course of a few minutes, freeze sea water of the specific gravity of 1026. The water congealed when the thermometer reached 27°; then it rose to 28°, and remained at that temperature. This experiment being repeated with another portion of the same water, but more slowly and with weaker sulphuric acid, the temperature gradually sunk to 20°, when the whole mass froze at once, the ice being quite smooth and not at all frothy, though it did not exhibit the dry snowy surface which is observed in the freezing of fresh water under similar circumstance. The thermometer, as in the former case, rose to 28° instantly, and

* Philosop. Trans. Vol. 66.

remained stationary at that point. The ice being taken out of the receiver and the vessel inverted, a small quantity of strong brine drained off from it. This was mixed with the portion of water, which, in the former experiment, had escaped congelation, and the specific gravity of these mixed unfrozen residues proved to be 1035,16; whilst that of the frozen portions, after being washed with distilled water, then melted and mixed together, was 1015,2. These results seem to show that a certain degree of rest and slowness in the process, and probably also a certain mass of water, are conditions required for the entire separation of the salt; and hence we are enabled to account for the slight differences which we observe in this respect, in various specimens of water taken from the frozen surface of the sea.*

With regard to the important questions connected with the temperature of the Arctic seas, it will be seen by a reference to the tables, that this interesting subject of inquiry was not overlooked in the late Northern Expeditions, and that some curious observations were made in those regions, on the temperature of the sea, both at the surface and at different depths. Thus for instance it is stated in the 1st. Article of Table I., on the authority of Captain Ross, that in lat. 66,30, and long. 68,30, west, the temperature of the air being 36°, that of the

* I have also frequently, in the course of these experiments, frozen, by means of cooling mixtures, small quantities of sea water in tubes, with a thermometer in the water. When a certain degree of agitation was used, the water generally froze at about 25° or 26°; but when, (as will be seen in the latter part of this paper,) a more considerable vessel of sea water was exposed to congelation, the vessel being quite full and kept at rest, the water was cooled to between 18° and 19° before it became solid. In either case the thermometer uniformly rose to 28°, at the moment of congelation.

water was found to be 38° at the surface; 30° at the depth of 80 fathoms; 29° at 200 fathoms; 28.5° at 400, and 25° at 670 fathoms.* These results are the more singular, as they are at direct variance with those obtained, nearly at the same period, by Lieut. FRANKLIN, in the Polar or Greenland seas, in higher latitudes. It will be seen by the curious and valuable table which Lieut. FRANKLIN has permitted me to annex to this paper, that, with only one or two exceptions, he uniformly found the sea to be sensibly warmer at great depths than near the surface, and that the difference often amounted to four or five degrees. Lieut. BEECHY, one of the officers of the same vessel, and Mr. FISHER, who was on board the Dorothea, both of whom made similar observations, have also favoured me with an account of their results, which, as will be seen by a reference to their respective tables, perfectly coincide, in their general import, with those of Lieut. FRANKLIN.

* Captain Ross in his account of a 'Voyage to the Arctic regions,' has himself published some of the results which he obtained respecting the temperature of the sea in Davis's Straits, and Baffin's Bay. Thus in latitude 72,22, longitude 79, he found the temperature of the bottom of the sea, at the depth of 1050 fathoms, 28,5° (Appendix, p. lxxxv.). And in latitude 72,23, having examined the temperature of the sea at the depth of 500, 600, 700, 800 and 1000 fathoms, he found that it gradually decreased from 35° to 28¼ (Appendix, page cxxiv.). These differences, though not so considerable as that above related, all concur in establishing the general fact, that the lower strata, in that particular track of the northern ocean, are colder than the surface. The instrument which Captain Ross employed, was a register-thermometer, the indications of which were occasionally compared with the temperature of the mud and earthy fragments of various kinds which he raised from the bottom of the sea, by an appropriate instrument of his own contrivance; as this mud, both from the quantity raised, and from the manner in which it was confined, retained its temperature for a sufficient length of time not to be materially altered on reaching the surface.

On the other hand, Lieut. PARRY, who had the command of the ship Alexander, in Captain Ross's expedition (and is now appointed commander of the second expedition to Baffin's Bay), fully confirms the observations made by Captain Ross, and also by Captain SABINE,* on board Captain Ross's ship; so as to place beyond all doubt the fact of Baffin's Bay being colder at the bottom than it is at the surface.†

But although these points may be considered as satisfactorily established, it must be admitted that the various modes

* Captain SABINE has been so obliging as to furnish me with a table containing some of his observations on the subject, which will be found in the Appendix.

† Captain PHIPPS also states in his Journal (Appendix, page 142), that he found the temperature in Baffin's Bay, at the depth of 680 fathoms, as low as 40°, the surface being 55°, and the air 66½°.

Other observers have obtained, in other seas, analogous results. Thus, DE SAUSSURE having examined with great care the temperature of the Mediterranean at various depths, found it in two different places to be 10,6° of REAUMUR's scale, or about 56° FAHR. at the depth of 900 and 1800 feet, the surface being about 71°; and he was induced to conclude that the temperature of the Mediterranean at great depths is uniform, and not likely to be affected by the vicissitudes of the atmospheric temperature, or by changes of season (Voyage dans les Alpes, III. § 1351 and § 1391).

M. de HUMBOLT, whose attention was often directed to this subject, makes the following curious observation. "In the seas of the tropics we find that at great depths the thermometer mark 7 or 8 centesimal degrees (or about 45° FAHR.). Such is the result of the numerous experiments of Commodore ELLIS and of M. PERON. The temperature of the air in those latitudes being never below 19° or 20° (or about 56° FAHR.), it is not at the surface that the waters can have acquired a degree of cold so near the point of congelation, and of the maximum of the density of water. The existence of this cold stratum in the low latitudes is an evident proof of the existence of an under-current, which runs from the poles towards the equator: it also proves that the saline substances, which alter the specific gravity of the water, are distributed in the ocean, so as not to annihilate the effects produced by the differences of temperature." ("*Personal Narrative of Travels*," English edition, Vol. I. page 63.)

in which the experiments were made, could not be relied upon as to perfect accuracy.*

It is obvious that these defects in the methods employed, though affecting the precision of the results, and rather tending to render them less striking, could not in the least degree invalidate the general conclusion, that in Davis's Straits, and in Baffin's Bay, the sea, at great depths, is considerably colder than at the surface; while to the east of Greenland, and in rather higher latitudes, the temperature of the ocean follows precisely the opposite law.

These various facts having an obvious and immediate connexion with the density of water under different temperatures, my attention was naturally directed to that circumstance in respect to sea water, which had not yet, I believe, been the subject of direct investigation. It had been long suspected, but was first established by Deluc, and afterwards correctly ascertained by Sir Charles Blagden, that water, in cooling towards the freezing point, ceases to contract when its temperature reaches about the 40th degree; but that, on the con-

* Captain Ross, who generally used a register-thermometer, might easily have detected, by a comparative observation, any material error made in ascertaining the temperature of the mud which he brought up by his apparatus; and as he appears to have occasionally availed himself of that mode of checking his observations, we may presume that his results were free from any considerable error. Lieut. Franklin, on the other hand, when he could not reach the bottom, and was therefore unable to make use of my machine, employed that used by Dr. Irving, consisting of a leaden cylindrical vessel with two valves; a convenient apparatus, but which, as I before observed, is liable to some inaccuracy. He sometimes also used a corked bottle, which he sunk to a great distance from the surface, and by means of which he obtained, doubtless, water from considerable depths; but it was obviously impossible to estimate with exactness the precise depth from which this water was procured, or the change of temperature which it had undergone in traversing the upper strata.

trary, it begins to expand, and continues to do so till it becomes solid, at which moment it undergoes a farther and much more considerable expansion.* The question which I was desirous of ascertaining was, whether the same, or any analogous law, prevailed in regard to sea water.

The mode in which I first attempted to decide this point, was simply by cooling sea water, by means of cooling mixtures, till it reached the freezing point, and ascertaining its specific gravity, at each degree of temperature, as it approached congelation. Researches of this description are liable to a variety of practical difficulties, which I could not altogether overcome by this method, and the results which I obtained, offered slight inconsistencies, which prevented my relying upon their strict accuracy.† Still however they uniformly led me to the conclusion, that the law of greatest specific density at 40°, did not prevail in the case of sea water; but that, on the contray, sea water gradually increased in weight down to the freezing point, until it actually congealed.

Soon afterwards I used another method, which afforded more precise, and, as far as I am able to judge, decisive results. Instead of weighing the water, I measured its bulk, under various temperatures, by means of an appropriate apparatus. A sketch of this instrument (which was executed by Mr. NEWMAN) is given in Plate XII., and an explanation is annexed, which supersedes the necessity of any farther de-

* Philosophical Transactions for 1788, page 143.

† In experiments of this kind it is always necessary to make an allowance for the contraction of the glass vessel, the effect of which is to produce an apparent expansion of the fluid contained in it. There are formulæ for this purpose, and in particular that derived from ROY's experiments, which was adopted in GILPIN's Tables. According to ROY, a vessel of glass of the capacity of 10.000.000, would enlarge, by 1° degree, to the capacity of 10.000.129.

scription. The general conclusion drawn from four experiments, the results of which did not essentially differ from each other, was, that if a vessel filled with sea water of the specific gravity of about 1027, and of any temperature above the freezing point, be gradually and slowly cooled, the water contracts in bulk; and that this contraction continues to proceed, though in a diminishing ratio, till the temperature has reached 22° of FAHRENEIT's scale. At this point the water appears* to expand a little, and continues to do so till its temperature is reduced to between 19° and 18°, at which point the fluid suddenly expands to a very considerable degree, shooting up with great rapidity, and forcing itself out at the open end of the tube. At the same moment the thermometer rises to 28°, and remains at that point. The liquid is now

* I say *appears*, because the rise of the column, occasioned by the contraction of the glass, may in part account for this effect. It would have been extremely difficult to have estimated this circumstance with precision in the above experiment, because the tube belonging to my apparatus was not perfectly uniform in its bore. But by ascertaining the capacity of a given portion of the tube, as well as that of the bulb of the apparatus, and calculating the contraction produced in the glass by a reduction of four degrees of temperature, I have been able to satisfy myself that the effect arising from this contraction could only produce about one half of the rise of the column observed in this experiment. So that it can hardly be doubted but that some expansion, however small in its amount, takes place in sea water when cooled from 22° to 18°. But I hope to be soon able to repeat the experiment in a more perfect manner, by a method similar to that employed, for an analogous object, by MM. DULONG and PETIT, and described in their excellent paper on the "Mesures des Temperatures, &c. 1818."

It may also be objected to this experiment, that the bulb has not its interior cooled uniformly, since the surface must be acted upon by the application of cold before the central parts. This is true to a certain extent. But from the great slowness of the experiment (which lasted about three hours at each time), this source of error is in a great degree avoided; and, that the greatest degree of cold actually reached the centre of the vessel, was proved by a nucleus of ice being formed in it, which closely invested the bulb of the thermometer.

found frozen, and in a few minutes the maximum of expansion is obtained. During this congelation the apparatus was never broken, and I satisfied myself by various trials with other vessels, that if a vent, however small, be allowed to sea water at the moment of freezing, the vessel is preserved entire, which, it is well known, scarcely ever happens in the case of common water.*

A singular consequence to be drawn from these experiments seems to be, that, since sea water does not begin to expand till it has been cooled below the point at which it usually freezes, if its congelation were not retarded, it would become solid without undergoing any previous expansion, and the law in question would altogether cease to exist in the case of sea water.

With regard to the singular anomalies of temperature in the Arctic seas, which have given rise to this digression, though some of the facts in question may now be more easily understood, it would be premature, until the observations have been multiplied, and the facts themselves more accurately investigated, to attempt to bring them under any

* The ice thus produced, it should be remembered, is very different from that which forms on the surface of the sea, since the latter parts with its salt in the act of freezing, a separation which can but very imperfectly take place in confined vessels. Accordingly I found the ice produced in this experiment soft and compressible like the water-ice of confectioners.

With regard to the quantity of expansion which sea water undergoes, in confined vessels, at the moment of freezing, I have been able to estimate it with ease, and with sufficient accuracy, by freezing a known weight of water in a phial, connected with an open tube, and ascertaining exactly the proportion of water forced into the tube during congelation. The result of two experiments which agreed perfectly with each other, was, that the expansion of sea water, when passing to the state of ice, is equal to 7,1 per cent of it bulk.

general law, or to explain the phenomena by particular theories.* Why, for instance, two neighbouring and almost contiguous portions of the ocean, placed nearly alike in regard to solar influence, should differ so widely in the temperature of their waters, the warmer strata being, in one case, found lying above the colder, while in the other that order is reversed, appears perfectly unaccountable. Whether, also, this singular circumstance may lead to inferences bearing upon the question now at issue respecting a north-west passage, I shall not presume to decide. But I may be allowed to indulge a hope, that the facts collected in this paper, may assist future inquirers in forming more accurate views of those grand phenomena of nature, in which the navigation of certain seas, the vicissitudes of seasons, and the geological history of the globe are so essentially concerned; or that they may at least be the means of inducing other and abler observers to turn their attention to this interesting subject.

* Count Rumford, in one of his Philosophical Essays (Vol. II. Essay VII.), in endeavouring to trace this class of natural phenomena to final causes, was led to some speculations and generalizations on the comparative temperature of the seas, and of large lakes, at their surface and at different depths, and on the relation which these temperatures bear to climate and to human comfort, which, however hypothetical, possess considerable plausibility and interest. Count Rumford's general idea was that the uniform temperature of large lakes at great depths, which De Saussure found in the Swiss lakes to be constantly between 41° and 42°, was naturally explained from the circumstance since discovered, of water possessing its greatest density at about that temperature; and he conceived that the object of this law of nature was to preserve in winter a store of warmth at the bottom of these lakes, by which their freezing was retarded at the surface, and altogether prevented at a great depth, thus affording a check to the effects of severe winters. With regard to salt water, however, he took it for granted that the law which fixes the greatest density at about 40°, did not prevail; but that, on the contrary, sea water being denser in proportion as it is colder, the coldest strata must occupy the bottom of the sea, while the warmest arising to the surface, serve to moderate the effects of the Arctic cold. He then

§ II. *On the Saline contents of the Waters of different Seas.*

I confined my remarks, in the first part of this paper, to the subject of the specific gravity and temperature of sea water, in various seas and in different latitudes. It remains for me to offer a few observations on the saline contents of these waters.

An accurate analysis of all the specimens which I have noticed in this paper would have been a most laborious, and indeed almost interminable undertaking, which would not have afforded any adequate object of curiosity or interest. All that I aimed at, therefore, was to operate upon a few of the specimens, so selected as to afford a general comparison between the waters of the ocean in distant latitudes and in both hemispheres, and to enable me also to ascertain whether particular seas differed materially in the composition of their waters.

For this purpose, availing myself of the experience I had obtained, in former inquiries of this kind,* respecting the

supposed that the colder and heavier strata would form sub-marine currents, constantly moving from the vicinity of the poles towards the equator, and occasioning upper and warmer currents precisely in an opposite direction. It is obvious that this theory, though capable of explaining some of the phenomena above mentioned, cannot apply to those of an opposite nature, also related in this paper. Yet these may possibly depend upon peculiar and local causes; and I cannot omit to observe, that M. De Humboldt, in the work already quoted, entertains notions of an exchange constantly going forward between the waters of the Polar regions and those of the Equatorial seas, which bear considerable analogy to those of Count Rumford, and cannot fail to give them additional weight.

* See an 'Analysis of the Brighton Chalybeate,' published in Dr. Saunders's Treatise on Mineral Waters, 1805. Also 'An Analysis of the Waters of the Dead Sea and

difficulty, and indeed the impossibility, of analyzing complex solutions of saline substances with a view to obtain a precise and certain knowledge of the state of combination in which the salts exist in these solutions, I contented myself with ascertaining, first, the proportions of saline matter yielded by a given quantity of each water, and afterwards, the proportions of acids and earths contained in these respective waters; thus presenting data which are quite divested of theoretical views, and from which the composition of those waters may at any time be inferred in the way which may be deemed most eligible.

It has been long known that the principal salts contained in sea water are muriate of soda and muriate of magnesia, and that it contains also sulphuric acid and lime. But whether these ingredients existed in the form of sulphate of soda, or of sulphate of lime, or muriate of lime, or sulphate of magnesia, was more or less a matter of conjecture, as the different states of binary combination which they assume, are modified during evaporation by the different degrees of solubility which the salts possess, and are liable to be influenced by heat and concentration, the very processes which are used in attempting to resolve the question. These difficulties have been ably discussed by Dr. MURRAY,* whose reasonings and experiments on the subject have given great plausibility to the doctrine which he has proposed, according to which the salts contained in sea water are supposed to be

River Jordan;' Philos. Trans. 1807. And 'An Analysis of an Aluminous Chalybeate Spring in the Isle of Wight;' Geolog. Trans. Vol. I. 1811.

* See 'An Analysis of Sea Water' read in 1816, and published in the Edinburgh Transactions, Vol. VIII. and also a 'Formula on the Analysis of Mineral Waters,' printed in the same volume.

Muriate of soda,
Muriate of magnesia,
Muriate of lime,
Sulphate of soda.

Still however, it must be admitted that a degree of doubt remains respecting the mode in which the sulphuric acid is combined, and that we can only pronounce with certainty upon the proportions of acid and base taken singly, as I have explained above. My experiments, therefore, were confined to the following points.*

1st. To ascertain the quantity of saline matter contained in a known weight of the water under examination, desiccated in a uniform and well defined mode; and to compare it with the specific gravity of the water.

2ndly. To precipitate the muriatic acid from a known weight of the water, by nitrate of silver.

3dly. To precipitate the sulphuric acid by nitrate of barytes, from another similar portion of water.

4thly. To precipitate the lime from another portion of water, by oxalate of ammonia.

5thly. To precipitate the magnesia from the clear liquor remaining after the separation of the lime, which is best effected by phosphate of ammonia, or of soda, with the addition of carbonate of ammonia.

The soda, by this method, is the only ingredient which is not precipitated, and which therefore, can only be inferred

* It is but just to mention that I received, in this part of the inquiry, much valuable aid from Mr. Willson, who has many years acted as assistant, to my colleagues and myself, in the Chemical Theatre of Guy's Hospital.

by calculation. But if the processes are conducted with sufficient care, this mode of estimating the proportion of alkaline muriates is susceptible of great accuracy, as I had an opportunity of ascertaining by some comparative experiments which I related at full length in the analysis of the waters of the Dead Sea.*

The whole of the results obtained by this mode of investigation, has, for the sake of brevity, been condensed into a table which is annexed to this paper, and upon which it is unnecessary to detain the Society by any farther comment. It will be seen by this table that, with the exception of the Dead Sea, and of the Lake Ourmia,† which are mere salt ponds, perfectly unconnected with the ocean, all the specimens of sea water which I have examined, however different in their strength, contain the same ingredients all over the world, these bearing very nearly the same proportions to each other; so that they differ only as to the total amount of their saline contents.‡

* In devising the above method, I followed, step by step, the plan which I had myself pointed out, and actually used, in various analyses, and particularly in that of the Dead Sea, and of an aluminous chalybeate, in the Isle of Wight, as may be seen by a reference to these papers. It is satisfactory to observe that Dr. Murray adopted, several years afterwards, from considerations of the same kind, a mode of proceeding precisely similar, and indeed that he proposed in a subsequent paper, a general formula for the analysis of mineral waters, in which this method is pointed out as likely to lead to the most accurate results. And this coincidence is the more remarkable, as it would appear, from Dr. Murray not mentioning my labours, that they had not at that time come to his knowledge.

† I had only between 2 and 300 grs. of water from this curious lake, which is so nearly saturated, that it begins to deposit crystals the moment that heat is applied to it. Though it contains no lime, it yields about 20 times as much sulphuric acid, and six times as much muriatic acid as sea water does, as may be seen by the annexed table. Dr. Wollaston has also detected traces of potash in this water.

‡ The Yellow Sea, in the Chinese ocean, has some peculiarities which deserve to

It would hardly be consistent with the plan which I have followed in this inquiry, to enter minutely upon the analysis of the waters of individual seas, since, instead of dwelling on analytical details, I have rather aimed at presenting an extensive and comparative view of the subject, for the purpose of drawing certain general inferences. Yet as my experiments were made with care, and appear from their consistency with each other, to justify some degree of confidence as to the accuracy of their results, it may not be out of place to select from the above table some individual water, with a view

be noticed. The smell of the specimens put into my hands by Captain Hall, was exceedingly *hepatic*, like that of a strong solution of sulphuretted hydrogen, and this water formed with silver a black precipitate. I twas clear and transparent; but had a greenish yellow colour. Nitric acid made it milky, and precipitated sulphur from it. When boiled it gave out sulphuretted hydrogen gas, and deposited a yellowish sediment, which proved to be carbonate of lime, in the proportion of 0.7 grs. for 500 grs. of the water, and without any sulphur being mixed with the sediment. The interior of the bottles was found blackened, so as to render the glass quite opaque; but the black film was easily wiped off, and the glass was not permanently stained. After evaporating the water to dryness, the residue dissolved readily in water, with the exception of the carbonate of lime above mentioned, and the solution now precipitated silver perfectly white. In other respects the saline contents of this water did not differ from those of other seas. Its specific gravity was low (1022.9), but the salts, with the exception of a small deficiency in the magnesia, were the same as usual. The water was first put by Captain Hall into a green-glass bottle; but it was, some months afterwards, transferred into several white-glass phials, having glass stoppers, all of which exhibited the appearance above described. There is something in this developement of sulphur in sea water which is by no means well understood. Of two specimens brought from the same spot, and by the same individual, I have sometimes observed that the one had a smell of sulphuretted hydrogen, while the other was perfectly free from it. In the former case the cork was commonly blackened and decayed. I therefore suspect that in some instances the cork gives the impulse to the formation of sulphuretted hydrogen; but in others, and probably in the Yellow Sea, this change is likely to be owing to the presence of some vegetable or animal matter, and its gradual action on the saline water.

to show how the various statements which it contains may be reduced to the form in which analytical results are usually expressed.

Thus, for instance, if we select the water marked No. 27, which was taken up nearly in the middle of the North Atlantic, and the specific gravity of which was 1028.86, 500 grs. of this water yielding 21.3 grs. of saline matter, dried at 212°, we shall proceed in the following manner:

The muriate of silver obtained from 500 grs. of the water being 42 grs., 100 grs. of which are equal to 19.05 of dry muriatic acid, the 42 grs. of luna cornea will be equal to 8 grs. of muriatic acid.

The sulphate of barytes obtained from a similar portion of water being 3.85 grs. dried at 212° = 3.74 grs. dried at a red heat,* 100 grs. of which contain 34 grs. of sulphuric acid, the quantity of dry sulphuric acid in 500 grs. of the water will be (100 : 34 :: 3.74 : 1.27) 1.27 grs.

The oxalate of lime, from a similar portion of water, being 0.8 grs. dried at 212°; and 100 parts of oxalate of lime so dried being = 0.314† of pure lime, the quantity of lime in 500 grs. of the water will be be 0.314 grs.

The phosphate of magnesia being 2.7 grs., 100 of which contain 40 of magnesia, the quantity of magnesia in 500 grs. of the water will be 1.08 grs.

It appears, therefore, that the quantities of acids and earths

* I found by a careful experiment, made for the express purpose, that 100 grs. of sulphate of barytes, dried at 212°, were reduced by a red heat to 97.2 grs.

† I obtained this result from a direct experiment, in which 24 grs. of ignited muriate of lime = 12.24 grs. of pure lime, gave 31.2 grs. of oxalate of lime dried at 212°. Therefore, 31.2 : 12.24 :: 100 : 39.23.

contained in 500 parts of this water, and estimated in their uncombined state, are as follows:

Muriatic acid, - -	8 grs.
Sulphuric acid, - -	1.27
Lime, - - -	0.814
Magnesia, - - -	1.08

It now remains to estimate, from the above data, the compound salts contained in the water, according to their most probable state of combination as before explained; and to infer the quantity of soda belonging to the same portion of the water, a question which cannot well be ascertained by a direct process. This will be effected in the following manner.

Muriate of lime is known to consist of 51 parts of lime, to 49 of muriatic acid.* Therefore the above 0.814 gr. of lime = 0.302 of muriatic acid = 0.616 gr. muriate of lime, free from water.

Sulphate of soda, in its dry state, consists of 56 parts acid, to 44 soda;† and therefore the above 1.27 gr. of sulphuric acid = 1.01 soda = 2.33 grs. dry sulphate of soda.

Muriate of Magnesia, in a state of dryness, consists of 58.09 parts of muriatic acid, to 44.91 of magnesia.‡ Therefore the 1.08 of magnesia are equivalent to 1.497 of muriatic acid (41.91 : 58.09 : : 1.497) = 2.577 of dry muriate of magnesia, in 500 grs. of the water.

* Scale of Chemical Equivalents.

† Scale of Chemical Equivalents; and 100 parts of crystallized sulphate of soda consists of, sulphuric acid 24.5; soda 19.5; water 56. The above 2.33 grs. therefore would amount to 5.3 grs. crystallized sulphate of soda.

‡ Scale of Chemical Equivalents.

We are now enabled to estimate the quantity of muriate of soda. For the quantities of the muriatic acid already assigned to the earthy bases, being as follows, viz.:

in combination with lime – 0.302 grs.
———— with magnesia – 1.497 } 1.799 grs., and

the total quantity of muriatic acid being 8 grs. there will remain 6.2 grs. of the acid in combination with soda. But dry muriate of soda consists of 46.6 parts of muriatic acid to 53.4 of soda;* and consequently the 6.2 grs. of muriatic acid = 7.1 grs. soda = 13.3 grs. muriate of soda.

It appears therefore that the saline and earthy substances contained in 300 grs. of the specimen of sea water under examination, taken in the uncombined state, are:

Muriatic acid –	8 grs.
Sulphuric acid –	1.87
Lime – –	0.314
Magnesia –	1.08
Soda – –	8.11†
	18.774

And the same saline ingredients, in their state of combination, and supposed free from water, will be

Muriate of soda –	13.3
Sulphate of soda –	2.83
Muriate of lime –	0.616
Muriate of magnesia	2.577
	18.823 grs.

* Scale of Chemical Equivalents.

† Viz. 1.01 with the sulphuric acid, and 7.1 with the muriatic.

This total amount, it may be observed, does not exactly correspond with the saline residue of 21,3 grs. obtained by evaporation from 500 grs. of the water; but it should be remembered that this residue was dried at 212° only, which, with some salts, produces a considerable difference. I thought it important to ascertain the amount of this difference by direct experiments; and I found that 100 grs. of muriate of lime dried at 212°, were reduced by ignition to 61,9; so that if 100 grs. of muriate of magnesia, dried at 212°, be supposed to be brought to a state of perfect dryness, they will be reduced to 52 grs. As to the muriate of soda and sulphate of soda, when well dried at 212°, they lose no sensible weight by being ignited.

Upon making due allowance for the moisture contained in the two earthy muriates, according to the estimates just mentioned, we shall find the above result altered as follows:

Muriate of soda	13,8
Sulphate of soda	2,33
Muriate of lime	0,975
Muriate of magnesia	4,955
	21,460

Which result closely corresponds with the saline residue obtained by evaporation, which was 21,8 grs.

It remains for me in concluding this paper, to communicate to the Society an interesting fact on the composition of sea water just discovered by Dr. Wollaston, and which it is no small gratification to me to think that the present inquiry has been the accidental means of bringing to light. As I was beginning the chemical part of this investigation, Dr. Wol-

LASTON put the question to me, whether it was not probable that traces of potash might be found in sea water? I answered in the affirmative, and thought the fact well worthy of investigation;* but as no one could be better qualified than the Doctor himself to put his own suggestion to the test of experiment, I supplied him with sea water, and begged of him to favour me with his results, which he has just communicated to me in a note to the following effect:

"The expectation which I expressed to you that potash would be found in sea water as an ingredient brought down by rivers from the decay of land-plants, is now fully confirmed by experiments on waters obtained from situations so remote from each other as to establish its universality.

"There is no difficulty in proving the presence of this ingredient by muriate of platina. For though the triple muriate of platina and potash is so soluble that this reagent causes no precipitate from sea water in its ordinary state, yet when the water has been reduced by evaporation to about $\frac{1}{8}$th part, so that the common salt is beginning to separate by crystallization, the muriate of platina then causes a copious precipitate.

"If this precipitate be mixed with a little sugar and heated, the platina is reduced, and muriate of potash may be separated from it by water, and the nature of its base shown by its yielding crystals of nitrate of potash with nitric acid.

"I evaporated a pint of the water which you sent me (marked No. 9, specific gravity 1026,22) taken up by Captain Ross in

* I, in my turn, put the question to Dr. WOLLASTON whether it was not probable that minute quantities of all soluble substances in nature might be detected in sea water?

Baffin's Bay, from the depth of 80 fathoms, latitude 86°,32′, longitude 76°,46′ west. When this had been reduced to about $\frac{1}{30}$th part, I drained the liquor from the salt that had formed, which I also washed with a little water, and by adding muriate of platina to the drained liquors, I had a yellow precipitate which weighed 12,4 grains.

As the fluid poured from this precipitate measured $\frac{1}{4}$ of a fluid ounce, I estimate that this would retain in solution about three grains of the triple muriate, and hence the whole amount must be taken at 15,4, which by former experiments I consider as equivalent to about 6,4 sulphate of potash,* or 3,5 potash.

Now, since the pint of water weighed about 7520 grains, $\frac{6,4}{7520}$ gives the proportion of potash about $\frac{1}{1200}$; but the quantity of mere potash is less than $\frac{1}{2000}$th part of sea water at its average density."

* Dr. Wollaston thinks it probable that the potash exists in sea water in the state of sulphate.

TABLE V. *Presenting a Synthetic View of the results obtained from the Analysis of different Seas; the quantity of water operated upon being in every instance supposed to be 500 grains.*

Description of the Specimens.	Specific Gravity.	Residue of Evaporation of 500 grains of water.	Muriate of Silver.	Sulphate of Barytes.	Oxalate of Lime.	Phosphate of Magnesia.	Total of Precipitates, from 500 grs. of water.	Observations.
		Grains.	Grains.	Grains.	Grains.	Grains.	Grains.	
Arctic Ocean. Spec. 1.	1027.27	19.5	39.7	3.3	0.83	2.7	46.55	The quantity actually operated upon was 500 grains.
Arctic Ocean. Spec. 12.	1019.7	14.15	27.9	2.4	0.7	1.8	32.8	From surface. Quantity operated upon 500 grs.
Arctic Ocean. Spec. 67.	1002.35	1.75	3.2	0.1	0.05	0.03	3.37	Sea ice water; Coast of Spitzbergen. Operated on 500 grains.
Arctic Ocean. Spec. 14.	1027.05	19.3	38.9	3.25	0.95	2.9	46.	From a depth. Operated on 500 grs.
Equator. Spec. 35.	1027.85	19.6	40.3	3.7	0.9	3.1	48.	From surface. Operated on 500 grs.
South Atlantic. Spec. 41.	1028.19	20.6	40.4	3.75	1.00	3.2	48.3	Operated on 250 grs.
White Sea. Spec. 58 and 59.	1022.55	16.1	32.8	3.0	0.6	2.2	37.6	Operated on 500 grs. but evaporated only 250 grs.
Black Sea. Spec. 56 and 57.	1014.22	10.8	19.6	1.95	0.55	1.5	23.6	Operated upon 500 grs. for the earths; but upon only 250 for muriate of silver and evaporation of the water.
Baltic. Spec. 60.	1004.9	3.3	7.	0.7	0.2	0.8	8.5	Operated upon 250 grains. All the precipitates were slightly tinged by some vegetable or animal matter.
Sea of Marmora Surface. Spec. 53.	1020.28	14.11	28.4	2.65	0.4	2.35	33.8	Entrance of Hellespont. Surface. Operated on 500 grs., except for muriate of silver.
Sea of Marmora Bottom. Spec. 52.	1028.19	21.	40.4	3.55	0.9	3.2	48.05	From the bottom. A little carbonate of lime was deposited during evaporation; but none from the water at the surface. Operated on 500 grs.
Middle of North Atlantic. Spec. 27.	1028.86	21.3	42.	3.85	0.8	2.7	49.35	Operated on 250 grs. for evaporation of the water and precipitation of muriate of silver. 500 for the other salts.
Yellow Sea. Spec. 48.	1022.91	16.1	32.9	1.35	0.75	2.2	37.2	During concentration deposited carbonate of lime. (See note, p. 195.) The water was yellowish, and had an exceedingly strong hepatic smell. Proportion of magnesia rather smaller than common. Operated on 500 grs.
Mediterranean. Spec. 51.	1027.3	19.7	38.5	3.6	0.8	3.0	45.9	From Marseilles, and therefore rather weak, from the vicinity of rivers. Operated on 100 grs. for evaporation and muriate of silver; and 250 for the other salts.
Dead Sea.	1211.	192.5	326.4	0.5	9.78	55.5	584.68	Philosophical Transactions, 1807.
Lake Ourmia, in Persia.	1165.07	111.5	237.5	66.0	0.	10.5	425.5	Specimen brought by the traveller Brown. Operated on 100, and 50 grs.

GENERAL OBSERVATIONS.

In the above experiments, the residues were dried as follows, viz. The residue obtained from the water by evaporation, was thoroughly dried at a boiling heat in a water-bath, till it entirely ceased to lose weight. The muriate of silver was heated to incipient fusion; the sulphate of barytes and oxalate of lime were dried at a boiling heat; and the ammoniaco-phosphate of magnesia, was heated to redness. No filters were used. The precipitates were washed, dried, and weighed, in the same glass capsules in which they were formed, with the exception of the magnesian salt, which was heated to redness by means of the blow-pipe, in a very thin and small platina crucible.

[To face page 202.

TABLE VI. *Showing the differences in Temperature of Water from a depth or bottom, and at the surface, observed on board His Majesty's Brig* TRENT, *in the Arctic Seas; by Lieutenant* FRANKLIN.

Date 1818.	Latitude North.	Longitude East	Water obtained either from a given depth or bottom, as expressed.	Its Temperature.	Temperature of Water at surface at same time.	Temperature of Air.	Remarks as to the Situation of the Vessel with respect Land or Ice.
May 26	76° 48′	12° 26′	Depth, 700 fathoms.	43°	33°	29°	The ice in small detached pieces around the vessel. The land of Spitsbergen distant 6 or 7 leagues. The temperature of the water obtained, was not tried until the bottle was taken below into the cabin, to which circumstance I think this extraordinary difference of temperature from that of the surface is to be attributed.
June 20	79. 58	11. 25	Bottom. 24 fathoms.	31	31½	30	Vessel closely beset by ice.
June 21	79. 56	—	Bottom. 19 fathoms.	31	30	30	Ship surrounded by ice.
June 22	80.	—	Bottom. 33 fathoms.	31	30	30	Surrounded by ice, not far distant from land.
June 23	79. 59	10. 12	Bottom. 21 fathoms.	32½	31½	30	Beset in ice, close to the land.
June 25	79. 51	10	Bottom. 60 fathoms. 17 fathoms.	34 34	33 33	34 34	In open water, near to the land. Clear of ice, about 6 miles from land.
June 26	79. 44	9. 33	Bottom. 15 fathoms. 34 fathoms.	34 34	34 34	35 31	In clear open water, some miles from the margin of the ice. Near to the land.
June 27	79. 51	10	Bottom 72 fathoms.	34½	34	36	Detached pieces of ice near to the vessel.
June 29	79. 51	10. 18	Bottom. 17 fathoms. 19 fathoms.	34 34	34 34	39 37	Near to the land, between two islands.
July 6	79. 48	10. 15	Bottom. 34 fathoms.	34½	34	36	Near to the land, passing between two islands.
July 8	80. 20	11. 30	Bottom. 120 fathoms.	36	33	35	Closely beset in ice—about 11 or 12 leagues from land.
July 8 P. M.	80. 20	11. 30	Bottom. 130 fathoms.	36½	31½	33	Closely beset in ice—muddy bottom.
July 9 P. M.	80. 26	11. 38	Bottom clay, 120 fathoms. 110 fathoms.	36 35½	31 30½	35	Beset as before—about the same distance from the nearest land.
July 10	80. 19	11. 24	Bottom. 119 fathoms.	36	32	—	Closely surrounded by ice.
July 11	80. 22	10. 30	Bottom. 120 fathoms.	36	32	40	Surrounded by ice—muddy bottom.
July 12	80. 20	11. 7	Bottom. 145 fathoms.	35½	32	36	Surrounded by ice—muddy bottom.
July 13	80. 22	11.	217 fathoms. Bottom.	37	32½	—	Rocky bottom.
July 13	80. 22	11. 2	235 fathoms. Bottom.	35½	32	40½	Surrounded by ice—rocky bottom.
July 13	80. 23	10. 55	Bottom. 237 fathoms. Muddy.	35½	31½	40	Beset amongst ice about thirty miles from land. A specimen has been forwarded to Dr. Marcet.
July 14	80. 26	—	Bottom. 233 fathoms. Muddy.	35½	32	39	Surrounded by ice.
July 15	80. 27 80. 28	10. 20 10. 20	Bottom. 198 fathoms. 185 fathoms.	36 36½	32 32½	38	Beset amongst ice.

June 21	79. 56	—	Bottom. 19 fathoms.	31	30	30	Ship surrounded by ice.
June 22	80.	—	Bottom. 33 fathoms.	31	30	30	Surrounded by ice, not far distant from land.
June 23	79. 59	10. 12	Bottom. 21 fathoms.	32½	31½	30	Beset in ice, close to the land.
June 25	79. 51	10	Bottom. 60 fathoms. 17 fathoms.	34 34	33 33	34 34	In open water, near to the land. Clear of ice, about 6 miles from land.
June 26	79. 44	9. 33	Bottom. 15 fathoms. 34 fathoms.	34 34	34 34	35 31	In clear open water, some miles from the margin of the ice. Near to the land.
June 27	79. 51	10	Bottom 72 fathoms.	34½	34	36	Detached pieces of ice near to the vessel.
June 29	79. 51	10. 18	Bottom. 17 fathoms. 19 fathoms.	34 34	34 34	39 37	Near to the land, between two islands.
July 6	79. 48	10. 15	Bottom. 34 fathoms.	34½	34	36	Near to the land, passing between two islands.
July 8	80. 20	11. 30	Bottom. 120 fathoms.	36	33	35	Closely beset in ice—about 11 or 12 leagues from land.
July 8 P. M.	80. 20	11. 30	Bottom. 130 fathoms.	36½	31½	33	Closely beset in ice—muddy bottom.
July 9 P. M.	80. 26	11. 38	Bottom clay, 120 fathoms. 110 fathoms.	36 35½	31 30½	35	Beset as before—about the same distance from the nearest land.
July 10	80. 19	11. 24	Bottom. 119 fathoms.	36	32	—	Closely surrounded by ice.
July 11	80. 22	10. 30	Bottom. 120 fathoms.	36	32	40	Surrounded by ice—muddy bottom.
July 12	80. 20	11. 7	Bottom. 145 fathoms.	35½	32	36	Surrounded by ice—muddy bottom.
July 13	80. 22	11.	217 fathoms. Bottom.	37	32½	—	Rocky bottom.
July 13	80. 22	11. 2	235 fathoms. Bottom.	35½	32	40½	Surrounded by ice—rocky bottom.
July 13	80. 23	10. 55	Bottom. 237 fathoms. Muddy.	35½	31½	40	Beset amongst ice about thirty miles from land. A specimen has been forwarded to Dr. Marcet.
July 14	80. 26	—	Bottom. 233 fathoms. Muddy.	35½	32	39	Surrounded by ice.
July 15	80. 27 80. 28	10. 20 10. 20	Bottom. 198 fathoms. 185 fathoms. Mud.	36 36½	32 32½	38	Beset amongst ice.
July 16	80. 26	11. 25	Bottom. 173 fathoms. Clay and mud.	36½	32	39	Closely surrounded by ice about 30 miles from land.
July 17	80. 27	11.	Bottom. 285 fathoms.	35½	34	—	Ice very closely besetting the vessel.
July 18	80. 26	10. 30	Bottom. 305 fathoms. Muddy.	36	32½	36	
July 19	80. 24	11. 14	Bottom. 103 fathoms.	36½	31½	41	The ice closely surrounding the vessel.
July 20	80. 21	10. 12	Bottom. 188 fathoms.	35½	32½	34½	More open water than usual, distance from land 10 leagues.
July 21	80. 14	12. 19	Bottom. 95 fathoms.	35½	32½	41½	Surrounded by ice.
July 22	80. 15	11.	Bottom. 83 fathoms.	35½	31	41	Beset by ice.
July 23	80. 15	11. 36	Bottom. 73 fathoms.	36½	32½	37	The ice opening a little.
July 25	80. 15	11.	Bottom. 94½ fathoms.	36	32½	34	The water more open than for the last fortnight.
July 26	80. 20	11. 25	Bottom. 55 fathoms.	36	32	36	Surrounded by heavy ice.
Sept. 10 P. M.	75. 14 75. 14	3. 53 3. 53	Depth. 756 fathoms. 756 fathoms.	36 36	35 36	37	In open water—several miles distant from the margin of the ice.
Sept. 24	66. 35	5. 35	Depth. 260 fathoms.	41½	43	44½	A bottle of this was preserved. The vessel completely in the open ocean, 300 miles from any land or ice.

JOHN FRANKLIN.

Table VII. *Temperature of the sea at the surface, and at different depths; as observed by Lieutenant* Beechey, *on board the* Trent, *in the late voyage to the Arctic Seas.*

Date.	Latitude. North.	Longitude. East.	Depth.	Temperature of bottom.	Temperature of surface.
	° ′	° ′	Fathoms.	°	°
May 26	76. 48	12. 26	700	45	33
June 21	79. 56	11. 96	24	31	31.5
22	79. 58	11. 14	30	31	30
25	79. 52	9. 57	60	34	32
26	79. 44	9. 34	15	34	33
July 4	79. 49	11.	35	34.5	34.1
7	80. 16	11. 5	34	34.5	34.5
9	80. 23	9. 50	120	36	30.3
12	80. 21	11. 11	140	36.5	30.5
13	80. 23	11. 3	237	37	32
15	80. 27	.....	185	36.3	32.7
16	80. 27	11. 5	173	36.5	32
17	Ditto	Ditto	200	35.5	32.5
18	Ditto	Ditto	331	35	32
19	80. 25	11. 14	103	36.5	31.3
20	80. 24	10. 5	108	35.5	31.5
21	80. 13	11. 14	95	35.3	32
Sep. 24	66. 38	5. 44	260	41.5	43.5

We invariably found the temperature of the water increase with a southerly gale, and decrease as we approached the ice. At Spitzbergen, in August, the flood tide which came from the southward, was 3° warmer than the ebb: the former being 37°, the latter 34.°

Table VIII. *Temperature and specific gravity of sea water at the surface, and at certain depths, as ascertained by Mr.* Fisher, *on board the* Dorothea, *during the late voyage to the Arctic Seas.*

Temperature of different depths compared with the surface.*								
			Below the surface.			Surface at the same time.		
Date.	Lat.	Long.	Depth in Fathoms.	Sp. Gr.	Temp.	Temp.	Sp. Gr.	Situation of the Ship.
					°	°		
July 1819.	Between 79° 50′ and 80. 14	About 11.° 90 E.	40	1.0275	35.5	31.8	1.0267	About 10 leagues distant from Spitzbergen. The ship in general closely beset with ice.
			60	1 0275	36	32	1.0112	
			100	1.0274	36.3	32	1.0106	
			124	1.0279	36.7	33.5	1.0263	
			140	1.0279	36.5	32	1.0255	
			188	1.0281	42.5	33	1.0245	
			304	1.0282	39	31	1.0086	

* The above are the means between those observations most to be relied on. The Specific Gravities were taken with great care while the ship was beset in ice, and had no motion, with an hydrostatic balance, made for me by Newman, of Lisle-street.

TABLE IX. *Representing the Temperature at the Surface in a series of latitudes, both going out and coming home;* such as observed by Mr.* FISHER.

	Lat.	Temp.	Temp.	Sp. Gr.	
		°	°		
Going out.	60	46.7	50.9	1.0276	Coming home.
	61	45.5	49.2	1.0276	
	62	45.6	46.1	1.0275	
	63	45.3	44.2	1.0276	
	64	45.	43.1	1.0275	
	65	44.9	42.7	1.0275	
	66	44.8	45.3	1.0275	
	67	44.7	45.3	1.0274	
	68	42.8	47.3	1.0275	
	69	40.5	48.6	1.0275	
	70	39.2	40.9	1.0275	
	71	37.9	36.8	1.0276	
	72	36.7	36.2	1.0276	
	73	38.8	35.6	1.0277	
	74	38.6	35.9	1.0275	
	75	37.5	35.8	1.0275	
	76	35.9	35.6	1.0274	
	77	31.5	33.9	1.0273	
	78	30.9	36.4	1.0272	
	79	31.9	36.6	1.0267	
	80	32.7	32.7	1.0267	
	81			1.0058	

* Each of these results is a mean between all those taken between each degree of latitude and the succeeding one; thus the temperatures annexed to 65°, which are 44.9° and 42.7°, are means between the observations taken between 65° and 66°. The specific gravities are means of those taken both going out and coming home; for the differences of specific gravity (which, when the ship was in motion, was observed by an Hydrometer, by TROUGHTON) were probably in a great degree occasioned by the unavoidable errors of observation.

Table X. *Comparative Temperatures of Sea Water at the Surface and at certain Depths, as ascertained on Board the* **Alexander**, *during the late Voyage to Baffin's Bay, by Lieut.* **Parry**.

Day of the Month.	Latitude. North.	Longitude. West.	Depth in Fathoms.	Temperature of Water.		Temperature of Air in the Shade.
				Below.	Surface.	
1818.	° ′	° ′		°	°	°
June 1	63. 50	55. 30	145	32	36	35½
July 18	74. 50	59. 30	197	29½	32	37
Aug. 14	75. 56	66. 31	200	30½	32	36
			422	29½		
Aug. 22	76. 33	77. 10	102	29½	32	36
Aug. 24	76. 22	77. 38	100	30½	31½	33
			240	29½		
Aug. 25	76. 8	78. 31	54 Bottom.	29½	32	31½
Aug. 29	74. 58	77. 42	170	31	36	34
Sept. 1	73. 38	77. 19	125	30½	35	36
Sept. 5	72. 39	74. 30	190	30½	35	39
Sept. 6	72. 22	73. 06	246	30	36	41
Oct. 27	61. 48	1. 52	473	47	49	50

W. Parry.

Table XI. *Comparative Temperatures of the Sea, at the Surface and at certain Depths, as ascertained on Board the* **Isabella**, *during the late Voyage to Baffin's Bay, by Captain* **Sabine**.

Date.	Lat. North.	Long. West.	Depth. Fath.	Temperature. Below.	Surface	Air.	Remarks.
	° ′	° ′		°	°	°	
May 23	59 44		80	37	39	40	No soundings, deep sea.
Aug. 3	75. 52	63	415	29	34	38	Soundings in 430 fathoms.
14	75. 50	66	422	29½	32	38	Soundings in 450 fathoms, mud.
—	—	—	200	30½	32	38	
24	76. 35	78	240	29½	31½	33	
—	—	—	100	30½	31½	33	
25	76. 8	78. 21	54	29½	32½	31½	Soundings in 56 fathoms.
29	74. 59	76. 37	170	31	36	34	Soundings in 170 fathoms.
30	74. 4	79	235	29½	36½	37	No soundings.
Sept. 5	72. 37	74. 6	190	30½	35	35½	Soundings in 190 fathoms.
6	72. 23	72. 55	246	30	36	37	No soundings.
7	72. 16	71. 18	1000	28½	35	33	Soundings in 1000 fath.
19	66. 50	61	100	30	33	35	
—	—	—	200	29	33	35	Soundings in 750 fathoms.
—	—	—	400	29	33	35	
—	—	—	680	25½	33	35	
26	65. 50	59. 30	310	29	34	36	Soundings in 370 fathoms.
Oct. 4	60	58	900	35½	40	37	No soundings.
27	61	7	470	47	49½	50½	No soundings.

Edward Sabine.

Explanation of Plate XI.

CC. The brass cylinder.

W. The valves; the one in dotted lines opening upwards.

T. String connecting the two valves, so that they open and shut together.

D. A string fastened at one extremity to the valve in F, passing over the three pulleys PPP, and having a weight suspended at its other extremity W, so that the weight keeps the valves forcibly open.

BS. Springs pressing on the back of the valves, in order to close them forcibly when the weight is removed.

SS. Side springs, also tending to close the valves.

LS. Lock-spring, or catch, to keep the valves fast when closed.

A. The wire by which the machine is fastened to the line.

N. B. The machine is here represented in its natural dimensions; but it would answer the purpose much better if it were made three or four times larger, and if its weight were proportionally increased.

Explanation of Plate XII.

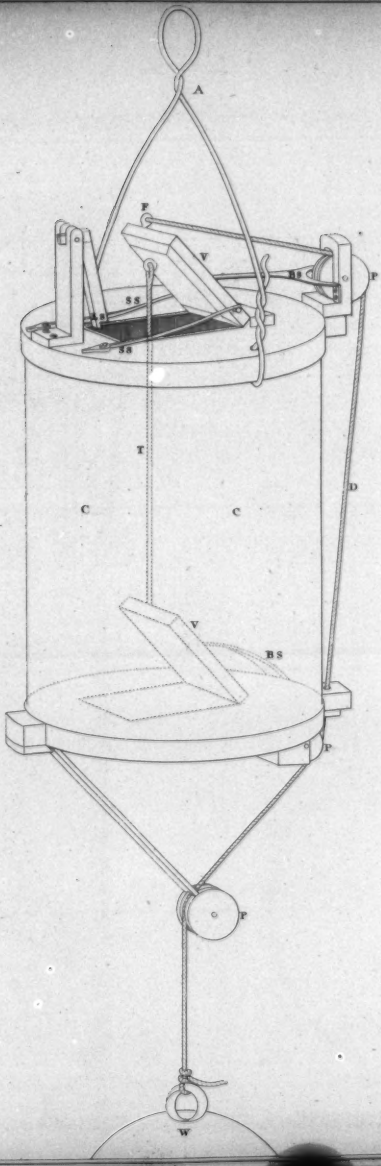
Fig. 1. The principle of this improved machine is essentially the same as that delineated in Plate XI, the valves VV, being kept open by means of a weight W, and closing themselves when the weight reaches the bottom.

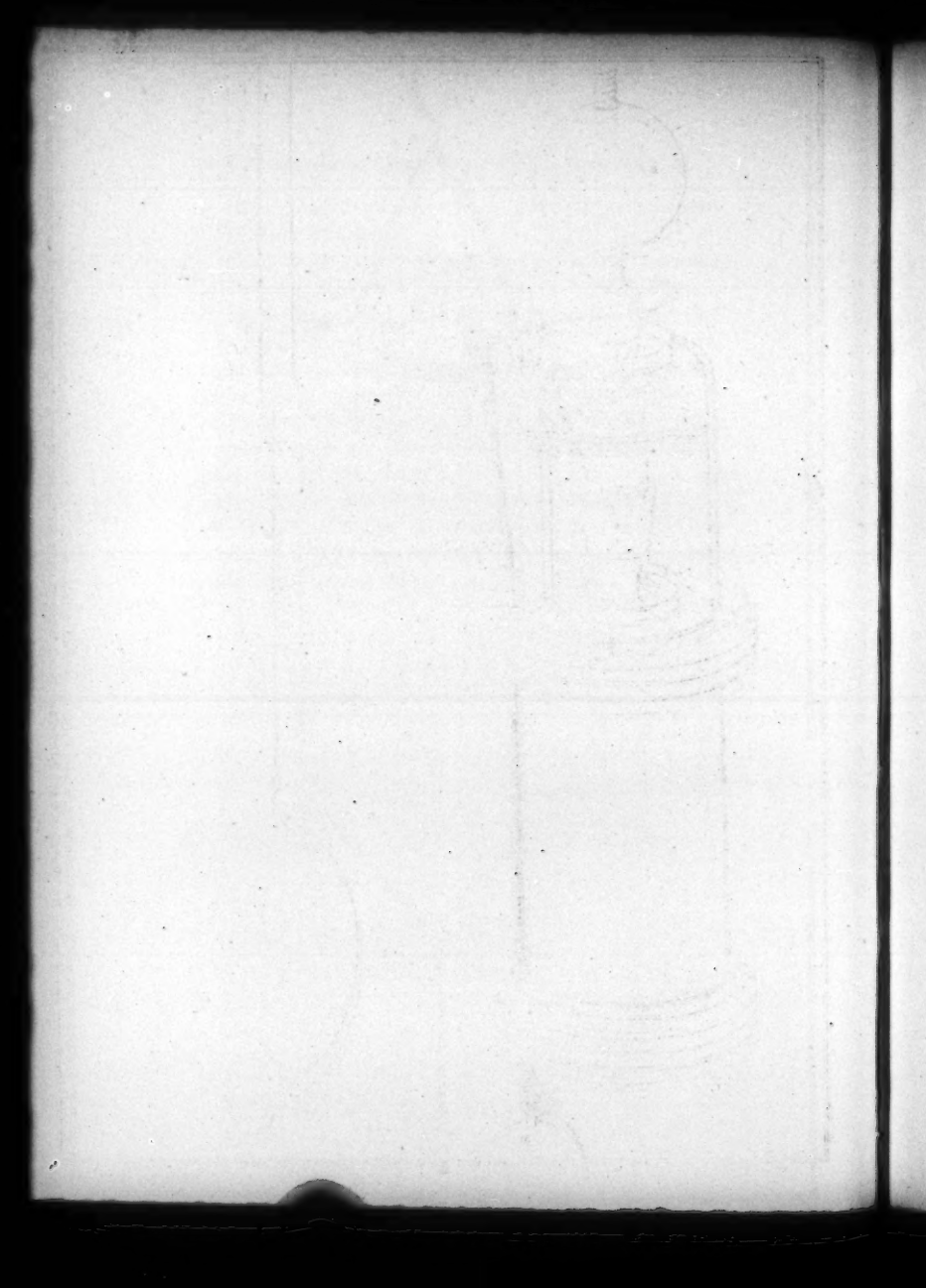
The valves in this machine are made of solid brass, and they fall by their own weight, so as to close the cylinder, the moment that the square FDE, which turns freely upon a pivot in P, is depressed in E, where it preponderates, the piece *cc*, which supports the valves, thus becoming unhooked from the recession of the hook, or clicket, in F. This may be effected in two ways; either by the weight W no longer pressing on the square in F, so as to keep it fast in its place, and therefore suffering it to recede, so as to disengage *cc*; or by letting down along the line a weight B, that shall fall upon DE along the rope A, and disengage the valves by the jerk it occasions. This constitutes the improvement by which water is now expected to be raised from any given depth, as well as from the bottom.

Fig. 2. This figure does not require any particular references. It represents the instrument in its natural size, which simply consists in a glass bulb of moderate thickness, capable of holding 844,6 grains of distilled water, with a neck or tube issuing from it, and containing a delicate mercurial thermometer, the elongated bulb of which is represented in dotted lines in the centre of the large bulb. To the end of this neck (the diameter of which is near half an inch), a long tube having rather a small bore, is ground air-tight, and

a scale of paste-board is fixed to it, in order to record the results of the experiment.

The bulb and neck being then entirely filled with sea water, and the tube fitted on, the fluid is thereby forced up into it to a certain height, which is marked on the scale. The bulb is now enveloped in cotton-wool, or any other bad conductor, and placed in a small jar, and this jar is immersed into a cooling mixture. The fluid is soon seen descending in the tube in proportion as the thermometer descends; and the gradual condensations of the water in the latter part of the experiment, such as they really occurred, may be seen marked on the scale. The level of the fluid in the tube is represented opposite No. 26, 25, 24, and 23, at which temperatures it remains stationary; and it then possesses the greatest specific density which sea water can attain.


A
F
V
BS
P
SS
SS
T
C
C
D
V
BS
P
F
W



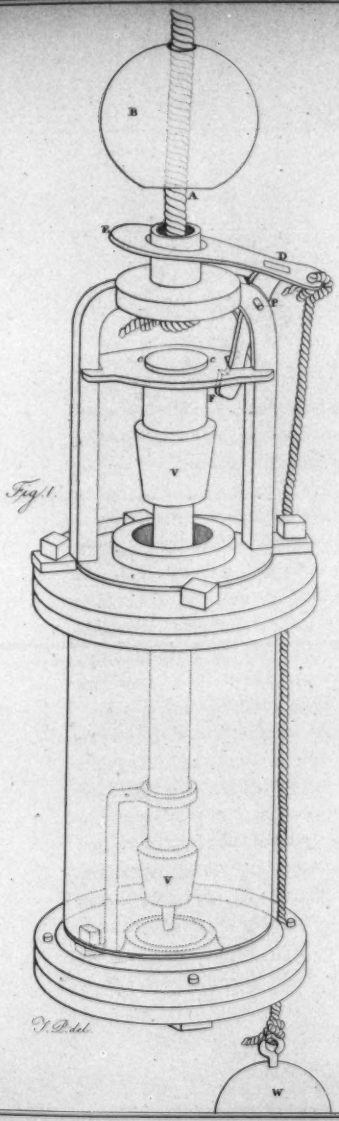


Fig. 1.

T. D. del.

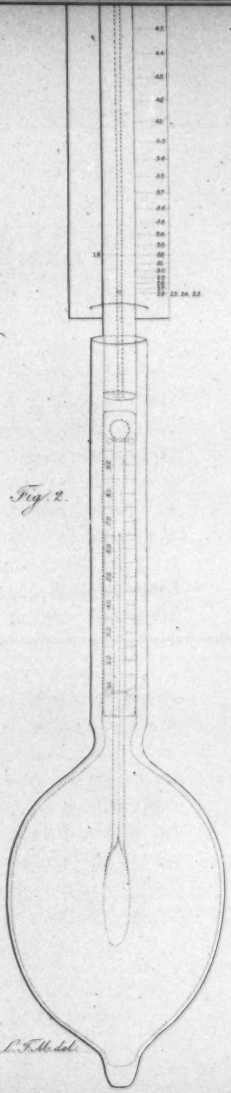


Fig. 2.

L. F. M. del.

J. Basire, sc.

XIII. *An account of the fossil skeleton of the Proteo-saurus.* By Sir Everard Home, *Bart. V. P. R. S.*

Read March 4, 1819.

In the year 1814, the skull and vertebræ of this fossil skeleton were first described in the Philosophical Transactions; and so much was the attention of the public called to the subject by that account, and so many specimens were brought under my observation, that in the year 1816, I was enabled to make many valuable additions to my former paper. In 1818, I laid before the Society the description of bones not before met with; and since that time, through the kindness of Mr. De la Beche, and Colonel Birch, I have procured materials, which put it in my power to describe nearly the complete skeleton, and to correct any errors, which the imperfect state of the first specimens had led me to commit.

One of these errors was, a belief that the orifice immediately before the margin of the orbit was natural, as it occurred in every specimen of the skull I had met with, five or six in number; but Colonel Birch has shown me a portion of a skull of very large dimensions, in which the nasal bones are perfect, and no such orifice is seen; so that the aperture described and delineated in my first paper, is the effect of injury the bones have sustained. A drawing of this skull is annexed, half the size of the original specimen. The drawing is made by Mr. Clift.

In a specimen in the possession of Mr. DE LA BECHE, the bones of the sternum are met with in their relative situation respecting the surrounding bones, affording a satisfactory proof that we are acquainted with all the parts of which the sternum is composed. In this specimen, the ribs can be traced to a greater extent than in any hitherto examined; they are not joined to the sternum by cartilages, but, as in the camelion and crocodile, are composed wholly of bone; and what is peculiar to themselves, each rib consists of one piece, having no intermediate joint: it describes a considerable curve in coming forward, and the outer side of the rib at that part is broader and stronger than any other. Their great length gives considerable depth to the chest.

In this specimen the bones are very small, but as they are completely formed, we must consider them to belong to a full grown animal.

When the vertebræ of the middle of the back in this specimen, are compared with those of the largest size that have been met with, it would appear that different species of the animal, were of very different sizes. In this specimen the diameter of a dorsal vertebra is only $\frac{4}{8}$ of an inch; in the largest that has been preserved it is 9 inches.

The drawing of this specimen is made by Mr. DE LA BECHE.

A specimen belonging to Colonel BIRCH, which in compliance with the wishes of my friend Mr. DE LA BECHE has been brought under my observation, contains nearly the entire skeleton of this extraordinary animal, and shows the important fact, that it had posterior as well as anterior feet; as it gives a posterior view, the bones forming the pelvis cannot be

made out, but these may be said to be the only ones with which we are now not acquainted.

This magnificent specimen is represented of its full size, in the annexed drawing, which has been made with so much ability by Mr. CLIFT, as to require no verbal explanation.

Since this Paper was laid before the Society, the author has received from Sir JOSEPH BANKS a specimen containing five fossil vertebræ of the Proteo-Saurus, each of them two inches in diameter, and one inch in thickness, found in the blue lias at Weston, near Bath, sent to him by the Rev. DANIEL LYSONS. This fact is important, as it proves that the fossil bones met with in the blue lias in that neighbourhood, which have always been considered to be those of the Crocodile, belong to this animal.

XIV. *Reasons for giving the name Proteo-Saurus to the fossil skeleton which has been described. By Sir* Everard Home, *Bart. V. P. R. S.*

Read April 1, 1819.

In the three Papers which I laid before the Society upon the subject of this fossil skeleton, I never ventured to hazard a conjecture upon the place in the chain of created beings, to which the animal belonged.

There were many circumstances which proved it to be unlike any animal at present in existence; some again making it an approach to the bird; others that connected it with fishes; so that I determined to prosecute the investigation till I had attained more satisfactory information respecting the skeleton, before I attempted to give the animal a name. This I think I have now done, the bones of the pelvis being the only ones not yet brought to light, and these are not necessary to enable us to make out the peculiar characteristics of the skeleton.

The discovery of the animal having four feet, established by the annexed drawings, removed it almost entirely from the finny tribe, in which there is no instance of such a mode of progressive motion.

It appears also distinct from the lacertæ, in which there is no instance of cupped vertebræ. All that tribe, as well as snakes and frogs, have the vertebræ united by regularly formed ball and socket joints.

These facts made it evident that the skeleton belonged to

an animal, somewhere intermediate between fishes and lizards, although belonging to neither; and the name Icthyo-saurus has been suggested by those who saw it in that point of view.

Finding the farther I advanced in my investigation, that the approaches to the lizard were greater and greater, and the only association with fishes was in the cupped vertebræ, I was led to examine the vertebræ of the Proteus, three specimens of which Sir HUMPHRY DAVY had just sent me from Germany, and found them all deeply cupped at both extremities, and the intervertebral cavity filled with a fluid. I found the same structure in the vertebræ of the syren from Carolina, and in those of the axolotl from Mexico, Dr. LEACH having placed at my disposal a specimen of the axolotl, in all respects similar to that brought to Europe by HUMBOLDT, and so well described by CUVIER. In both of these last species the cavity was filled with elastic ligament, and in the axolotl, the septum between the two cups was not completely closed.

Mons. CUVIER, who has proved in so satisfactory a manner that the proteus and syren are completely formed animals, has expressed his doubts respecting the axolotl; and hints at its resembling the larva of the salamander; but leaves the matter open for future enquiry.

When it is mentioned that the salamander has ball and socket joints to its vertebræ, and those of the axolotl are cupped, that celebrated anatomist will agree that these animals belong to different genera; and admit that, if the axolotl is a larva, the complete animal must have cupped vertebræ, which structure, I believe, is only met with in the proteus,

the syren, and the axolotl; and these three when compared together, appear to be equally complete animals.

This opinion is strengthened by the observation, that the parts of the rana paradoxa which are removed when it becomes a frog, contain no bone, all the tail beyond the pelvis being soft cartilage. The same remark holds good with respect to the larva of the salamander, and I should believe with all other larvæ.

From this statement it appears, that the proteus from Germany, the syren from Carolina, and the axolotl from Mexico, not only agree in having lungs and gills, and therefore capable of breathing both in air and water; but in having feet, and cupped vertebræ, and therefore capable of employing both the mode of progressive motion of land animals and of fishes; and whatever variations there may be among themselves, yet as they all possess these two great distinguishing characters, which no other animals have, they must be allowed to form a distinct tribe, or more properly a distinct class, which, not to multiply terms, I shall call Proteus, till a more appropriate name is given.

The fossil skeleton resembles the Proteus tribe in having feet and cupped vertebræ, but differs from it in having long ribs attached to a regularly formed sternum, admitting of the chest being very capacious, and also in having no arches fitted for gills; it cannot therefore be called a Proteus, although allied to it, in having two modes of progressive motion. It resembles the lacerta in its mode of breathing, but differs from it in the mode of setting on the ribs on the spine, the form of the legs and feet, and the bony plates of the eye balls; it cannot therefore be called a lizard.

Its place in the chain of animal creation is clearly pointed out to be between the proteus and lizard, and will be sufficiently marked out by calling it Proteo-saurus.

Explanation of the Plates.

Plate XIII. The representation of a portion of the skull of the Proteo-saurus, half the natural size, showing the form of the nasal bones immediately before the orbit. This is the only specimen in which these bones have been met with entire.

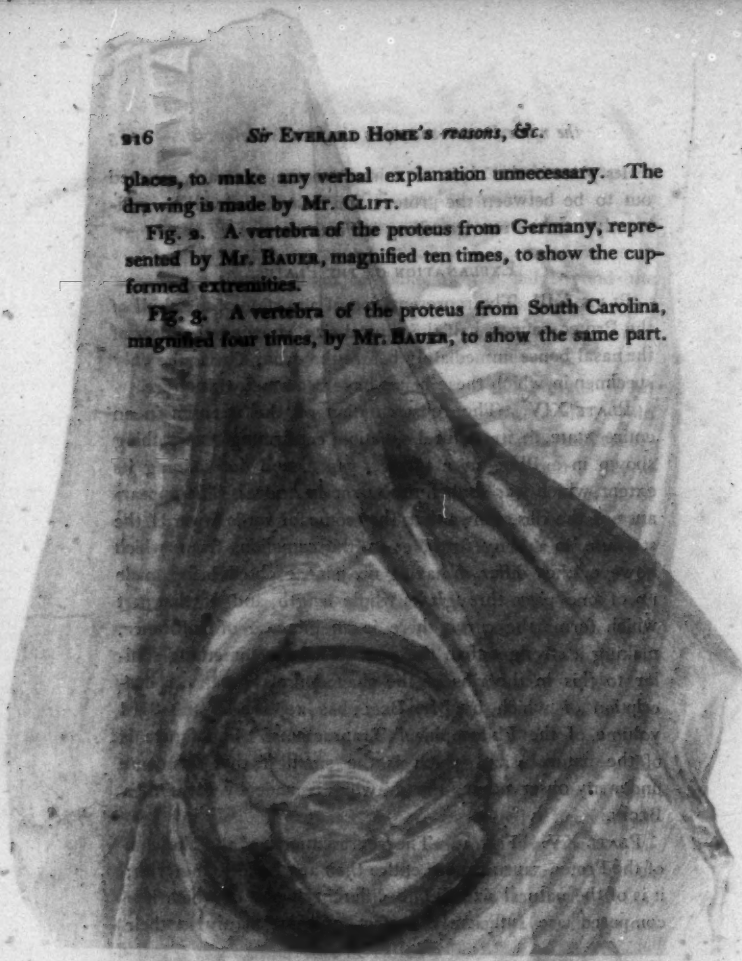
Plate XIV. The representation of the sternum in an entire state, in its natural situation, confirming every thing shown in a Plate in a former paper, and determining its extent, which was not before so exactly known. The appearance of the ribs, shows that they come forwards towards the sternum in a bony form, as in the camelion, from which however they differ in having no joint, each rib being made up of one piece through its whole length, and at that part which forms the curve there is an increase of substance, making it stronger than the rest. There is something similar to this in the ribs of the chætodon of Sumatra, a description of which, by Mr. Bell, has a place in the 83rd volume of the Philosophical Transactions. The figure is of the natural size, which is the smallest that has come under my observation; the drawing is made by Mr. De la Beche.

Plate XV. Fig. 1. The representation of the skeleton of the Proteo-saurus, more entire than any hitherto met with; it is of the natural size. The different bones of which it is composed are sufficiently perfect, and sufficiently in their

places, to make any verbal explanation unnecessary. The drawing is made by Mr. CLIFT.

Fig. 2. A vertebra of the proteus from Germany, represented by Mr. BAUER, magnified ten times, to show the cup-formed extremities.

Fig. 3. A vertebra of the proteus from South Carolina, magnified four times, by Mr. BAUER, to show the same part.



De la Beche. del.
J. Basire sc.

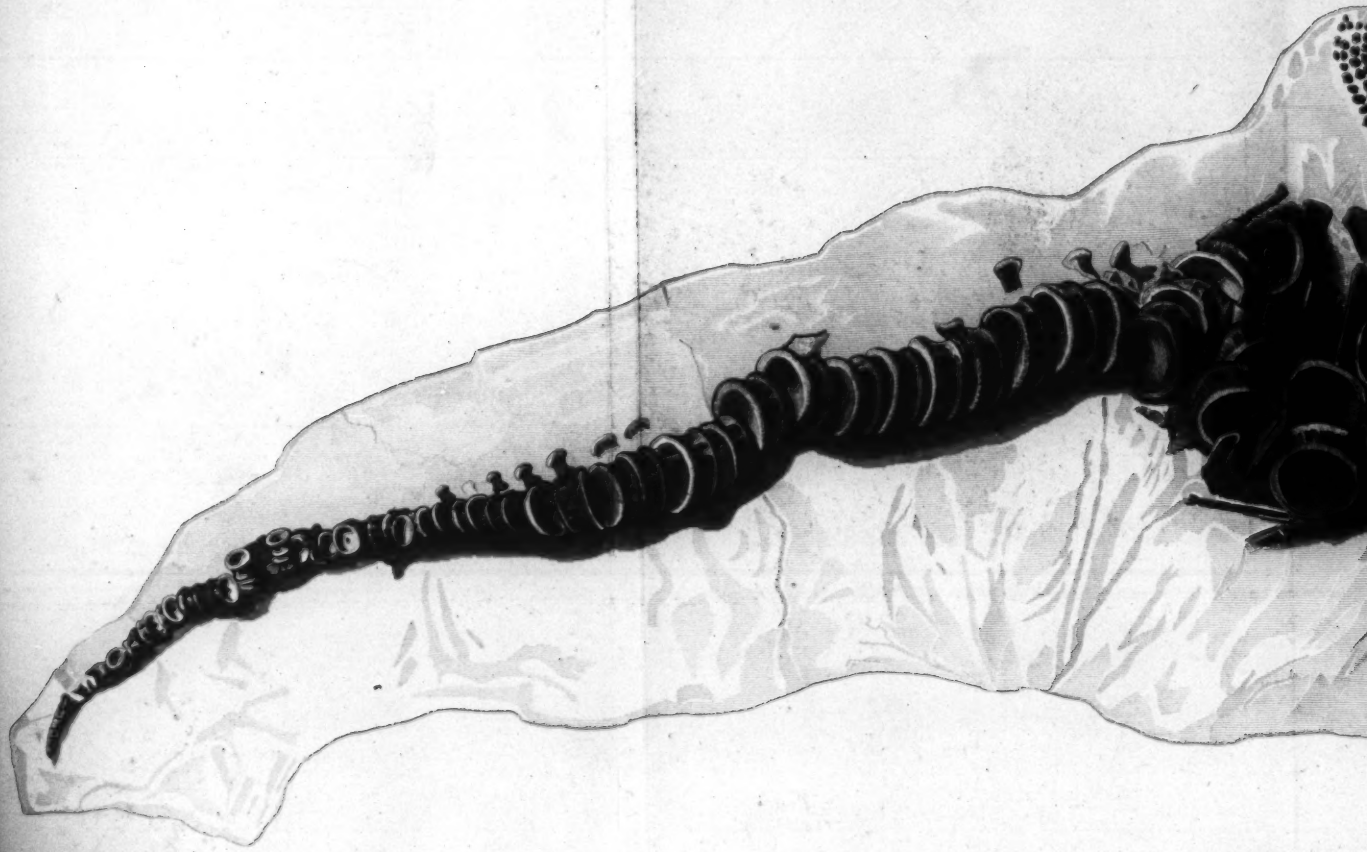


Fig. 1.

2

Proteus Carniola 10 diameters.

3

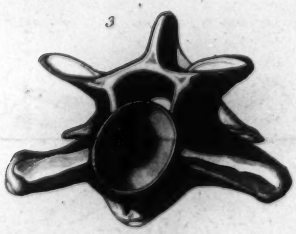
Proteus Carolinæ 4 diameters

Proteosaurus.

Fig. 1.

2

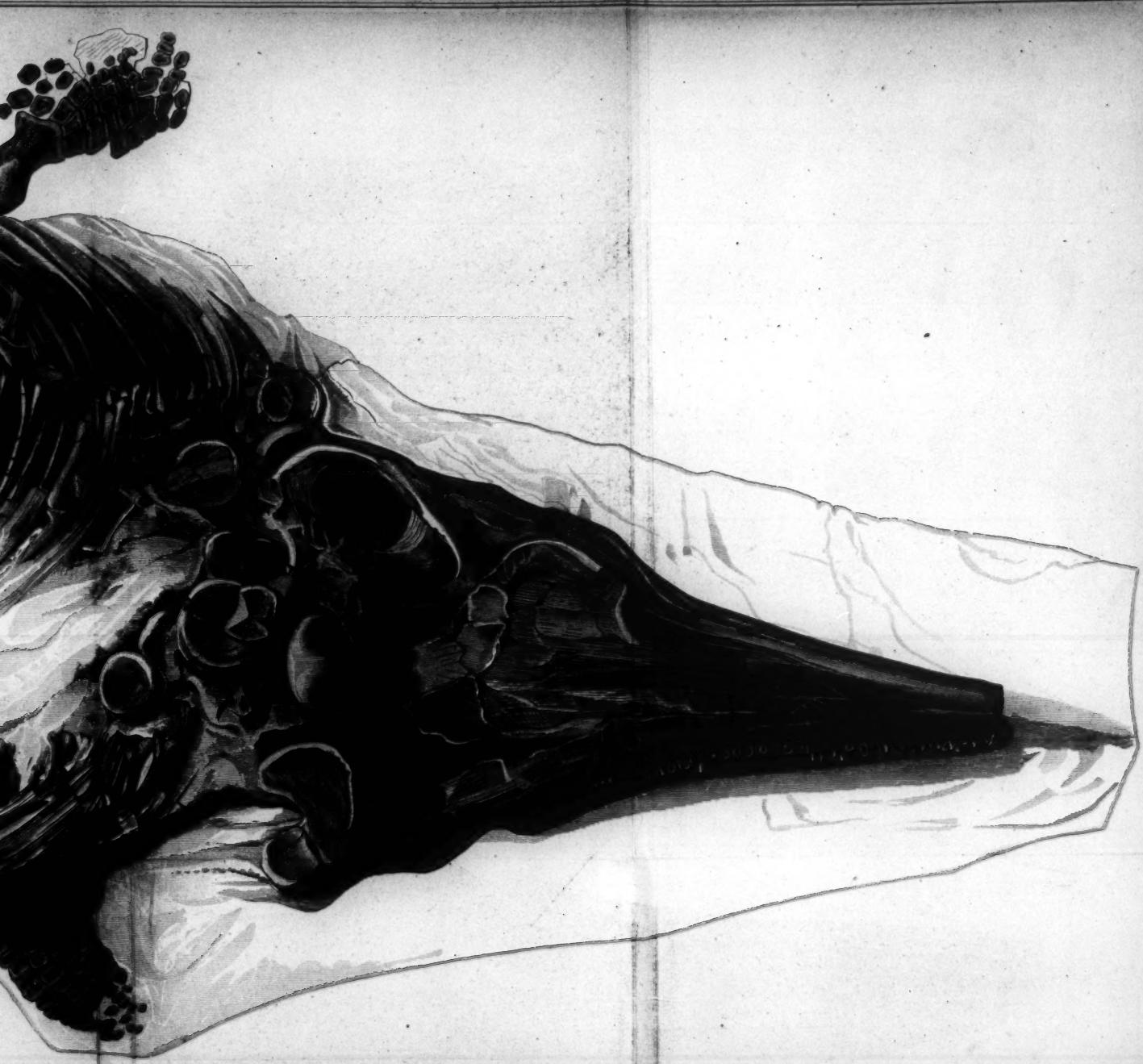
Proteus Carniola 10 diameters.



3

Proteus Carolina 4 diameters

Proteosaurus



XV. *Some Observations on the peculiarity of the Tides between Fairleigh and the North Foreland; with an explanation of the supposed meeting of the Tides near Dungeness.* By James Anderson, *Captain in the Royal Navy. Communicated by the Right Hon. Sir* Joseph Banks, *Bart. G. C. B. P. R. S.*

Read March 25, 1819.

HAVING observed that several Charts and Books of Navigation assert, that the tides from the North Sea and the Channel, or the Eastern and Western tides, meet in the vicinity of Dungeness and Rye harbour; and that, on such authority, this opinion has been too generally adopted by those, who have not had the opportunity or the inclination of making personal observations; as well as by the pilots on this part of the coast, who from being incapable, for the most part, of making observations or deducing inferences from facts before them, readily embrace the first theory they meet with *in print*, however erroneous or inconsistent; I have been induced to bestow all the attention in my power to the phænomena of the tides between Fairleigh and the North Foreland, and now venture to submit the result of my observations to the notice of the Royal Society. From having cruised constantly within these limits for nearly two years and a half, I have had many opportunities of making observations; but I must, nevertheless, profess my readiness to admit any alteration or improvement which may be pointed out by those more conversant with the subject; truth alone being the object of my enquiries.

I. *Phænomena of the Tides between Fairleigh and the North Foreland on the English coast, and Cape D'Alprèe and Calais on the French coast.**

The tides rise between the easternmost point of Fairleigh and the North Foreland from seven to eight feet higher than on either side of these points; and during the last three hours and a quarter in which the tides run to the eastward, the water *falls by the shore*, making it half tide of ebb on the shore, or by the ground, when the current of the tide changes and begins to run from the eastward to the westward; and it still continues to fall by the shore for two hours and three quarters after the tide has so changed; at which time it is *low water* every where within these limits. The course of the tide continues to run to the westward two hours and three quarters longer, during which time the water gradually rises by the shore, making nearly *half flood* by the land, at the time the current of the tide ceases to run to the westward; and returns again to the eastward, and continues to rise for three hours and a quarter, when it is high water by the ground. It then begins to fall again during the last three hours and a quarter, whilst the current of the tide sets to the eastward, as above stated; and so on in continual rotation.†

These appearances have, no doubt, given rise to the

* In the detail I shall principally confine myself to the English coast, as the phænomena, and their causes and effects, on the French coast, within the same limits, are precisely similar.

† Every one who has attended to the tides, in general, knows that, where there are no local obstacles to prevent it, they flow regularly about six hours in one direction and then make high water, and ebb about the same time in a contrary direction, then making low water.

erroneous opinion that the tides meet between Dungeness Point and Rye Harbour; but the real cause of these states of the tides within the particular limits I have described, seems to me to be the very great and sudden contraction of the channel between Dungeness and Cape D'Alprèe, and the South Foreland and Calais Point. In that part it becomes, all at once, narrower by more than half of its width to the eastward or westward of these points. Dungeness is a long narrow point which projects from Winchelsea on the west side, and Hythe on the east side, to the extent of nine or ten miles, at least, directly into the sea across the channel; and forms two deep bays, one on each side. Opposite to Dungeness is Cape D'Alprèe on the French coast, jutting out also into the sea, so as to contract the distance between it and the opposite point to about twenty-four miles, and this cape has also a large bay on each side, of which Boulogne bay is to the eastward. (See the charts.)

The distance from the South Foreland to Calais is only eighteen or nineteen miles, and between these opposite shores lie the Ridge, the Varn, the Goodwin, and several other shoals on both sides of the channel, all of which serve to contract this strait still more. The western tide, therefore, coming up the English channel, meets with a resistance to its course at Dungeness and Cape D'Alprèe, by the very sudden contraction of the space between these points; where, from the passage being insufficient to discharge the quantity of water brought from the westward, it must necessarily accumulate, until it encreases the channel both by deepening and widening it, so as to become adequate to the discharge of the body of water supplied by the impulse of the tide.

This accumulation is of course the same every where within the straits of Dover (from Dungeness to a ridge of rocks to the eastward of the South Foreland), and also extends some distance without them, as far as the easternmost point of Fairleigh on the one side, and the North Foreland on the other; Dungeness west bay, and the Downs, forming as it were two large natural basons or reservoirs at each extremity of these straits, for the reception of the accumulated water, until it can find a passage. On this account the water must rise accordingly by the ground, or on the shore, during the time of this accumulation, wherever it takes place; and it is indeed found to be at its greatest height or making *high water by the ground*, at about three hours and a quarter after the tide of flood has run from the westward. At this time all the sands without the North Foreland are covered, and afford a greater vent for the discharge of the accumulated water. The extensive flats also on both sides of the channel, on which the sea now flows in like a torrent, demand a greater supply than is received through the Dungeness passage. From this period then (viz. three hours and a quarter after the regular flood tide has run to the eastward), the water is drawn off from the places where it had accumulated, and begins to fall gradually by the shore during the remaining three hours and a quarter, in which the current of the tide runs to the eastward; making *half tide of ebb* by the ground, within the straits of Dover and the two reservoirs or basons, when the current of the tide ceases to run to the eastward; at which time it is *high water* every where without these limits, allowing for the inequalities of the coast, the water having now generally acquired a level.

When it is high water without the North Foreland, as at Margate, the Kentish Knock, &c., and the tide, which is the true or regular ebb tide, returns to the westward through the Downs, the water still continues to fall within the Foreland, and on to the easternmost point of Fairleigh, for two hours and three quarters of the first of the true or regular ebb tide; because the tide is falling generally, and the passage by Dungeness discharges the quantity brought by the ebb tide during that time. But when the true or regular ebb tide has run two hours and three quarters, it is *low water* by the shore, between the North Foreland and Fairleigh; because the channel through the straits of Dover, (becoming again too contracted to give vent to the great body of water which now presses forward from the Medway and the North Sea, augmented by the currents and tides discharged from the great continental rivers and inlets,) now again accumulates in the narrow passage, and in the Downs, from the North Foreland, and thus begins, from the above stated period, to rise by the shore.

It thus continues to rise for the remaining two hours and three quarters, at which time the true or regular ebb tide has ceased to run to the westward, and it is *low water* every where without the North Foreland, and to the westward of Fairleigh. But within these limits (viz. between the North Foreland and Fairleigh), it is *half flood*, in consequence of the accumulation of the water during the latter part of the ebb tide. The true or regular ebb tide, or tide *to* the westward, now ceases to run, and the true or regular flood tide *from* the westward returns, bringing with it a greater quantity or body of water than the Dungeness passage can yet admit,

consequently it must accumulate in Dungeness west bay, and rises proportionally from thence along the shore to the North Foreland for three hours and a quarter of the first of the regular flood tide, because the tide is rising generally every where; and about this time the channel, becoming broader and deeper by the accumulation of the water and rising of the tide, is again sufficiently large to discharge the supply. The accumulated water being thus drawn off, as before mentioned, and with an accelerated current, to cover the flats and fill up the Medway, and the continental rivers, again begins to subside by the shore, at which precise period it is *high water* by the ground within the limits of accumulation, both on the English and French coasts; but without these limits it is only *half tide of flood;* and therefore the true or regular flood tide must run three hours and a quarter longer to the eastward, during which time the water falls by the shore, within the limits of accumulation, until it finds its level every where; and so rises and falls in perpetual rotation.

The tide, within the limits where the water accumulates, is found to rise from 28 feet to 30 feet perpendicularly, which is from 7 to 8 feet higher than it generally rises in the Channel. The following seem to be causes of this extraordinary rise. At half tide by the shore, within these limits, the water has found its level every where, and half the rise of the tide here, at high water (viz. 14 or 15 feet), being drained off to make high water without the North Foreland, and produce the level, is now contained in a space twice the breadth it formerly occupied; of course it follows, the same quantity of water will only be half the depth (or from 7 to 8 feet) that

it was when confined in half the space it now occupies. It may hence be inferred that the rise of the tide here, more than elsewhere, is nearly equal to about one quarter the rise of tide, whatever it may be; but as this must always depend upon local circumstances, as the same effects could not be produced if the situation was different, no general reasoning can apply. It has also been ascertained, that the true or regular flood tide runs six hours and an half to the eastward, while the true or regular ebb runs only five hours and an half to the westward; which makes the current of the tide run an hour longer to the eastward than the westward; but I have always found, from actual observation, that these tides are very much influenced by the winds.

Upon the whole, however, from the easternmost point of Fairleigh to the North Foreland, on both sides of the channel, it is always *high water by the ground*, when the true or regular flood tide has run three hours and a quarter from the westward; always *half ebb by the ground*, when the true or regular flood tide ceases to run from the westward; always *low water*, when the true or regular ebb tide has run two hours and three quarters from the eastward; always *half flood tide by the ground*, when the true or regular ebb tide ceases to run from the eastward; and always *high water by the ground again*, when the true or regular flood tide has run three hours and a quarter from the westward, or nearly so, and so on continually.

II. *Meeting of the Tides near Dungeness.*

Although the foregoing observations may not decidedly prove that *the meeting of the tides* cannot take place at or near

Dungeness, yet I trust, that they rationally and intelligibly account for the peculiar phænomena of the tides which occur there, without attributing them to the *meeting* of the tides, which could never produce these appearances. But if the tides do meet at Dungeness, they must meet in a line directly across the channel; for it is a fact so well established, that no one I believe has ever ventured to contradict it, that the western or true flood tide makes high water at Beachy Head, Fairleigh, Dungeness, and Deal, at nearly the same time as at Dieppe, the Soame, Boulogne, and Calais, the opposite points on the French coast, each to each; and that the eastern or regular ebb tide makes low water, at the same time, at the same places.

But if the tides met in a line across the channel, it must be evident to every one who has been at sea, that such a meeting would occasion so tumultuous a war of elements, between two large bodies of water impelled against one another, by the current of the tide and force of the winds, at a velocity of from four to six miles an hour, according to the age of the moon and strength of the wind, as would produce, from their furious and violent concussion, so great a sea, that no ship could venture to encounter it without the most imminent danger.

That this is not the case, experience daily proves; and therefore the absurdity and fallacy of the doctrine which asserts it, are obvious. But every master of a collier or coasting vessel, trading from the northward to any western port, as Portsmouth, Plymouth, &c., knows that the flood tide sets from the northward and eastward, along the English coast until he gets as far as the sand called the Kentish Knock:

and if he can reach it by high water, he calculates rightly, that he will have an ebb tide thence, which will carry him to the westward for nearly six hours longer. From this it is evident that the tides from the northward and eastward, and southward and westward, both meet at the Kentish Knock,* as they both make high water about the same time at the same point; and then the ebb tide recedes from this point in the opposite directions to which the flood had advanced. The formation of the coast too, by gradually altering the course of the flood tide between the South Foreland and Buoy of the Nore, from E. N. E. to W. N. W. within the stream of the Goodwin Sands (while without this sand it continues to run E. N. E. and easterly), in a great measure prepares for their meeting, without that wild commotion and furious contention which their coming together in a directly opposite line across the channel, would inevitably occasion. It also admits of their gently blending their waters together, and smoothly taking the same course, along both sides of the Long Sand, &c. the one, viz. the flood tide from the eastward up the King's Channel into the Thames, and the other (the flood tide from the westward through the Downs) up the Queen's Channel into the Medway, making only a strong eddy or whirlpool about the Knock, and a foamy rippling where they meet, as they proceed onwards together.

But, although the tide from the northward and eastward makes flood tide along the N. E. coast of England to the Kentish Knock, yet it is equally well ascertained, that the tide from the southward and westward makes flood tide along the opposite coasts of Flanders, Holland and Jutland,

* This sand is of a circular shape, so formed by the continually whirling eddy of the tides.

as far as the entrance of the Sleeve. From this last mentioned fact it evidently appears, that the flood tide from the westward forms two distinct branches at the Kentish Knock, taking different directions; the smaller of which, consisting of the stream of the tide *within* the Goodwin Sands, takes its course W. N. W. up the Queen's Channel, as before stated; whilst the larger, consisting of the stream of tide *without* the Goodwin, continues its course E. N. E. and easterly along the Flemish and Dutch coasts, until it is lost near the entrance of the Sleeve, in the great body of tide from the northward and eastward.

The opposite tides which meet in the North Sea do not meet in a line directly across any part of it, but in a *diagonal line*, extending from the Kentish Knock to the entrance of the Sleeve; where there is no tide, but a strong current, which almost always sets from the Jutland to the Norway side in the Sleeve; and which most probably proceeds from the eddy, produced by the great body of water coming round the Naze of Norway, meeting the remains of the western tide, aided by the reaction of the Jutland shore. In fact, there is hardly any tide observable between the Horne reef and the entrance of the Sleeve.

The tides thus meeting in a diagonal line in the North Sea, gently and gradually blend their waters together, without causing the least tumultuous appearance, exhibiting merely a little foamy rippling, which can be discerned in fine weather only, when the general mass of water is perfectly smooth.

To prove farther that this is the nature of the tides in this part of the North Sea, let a ship, for instance, sail from North Yarmouth, or Harwich, for the Texel or Flushing on

the opposite coast, with the wind from the north-eastward, so that she can lay her course on the *larboard tack*; the pilot will prefer getting under weigh at *high water*, on the English coast, to take the ebb tide under his lee; and if he can get half channel over during the ebb tide, or by the time of low water on the English side, he will find a flood tide from thence, setting along the opposite coast, which will also set under his lee for six hours longer, running in the same direction as the ebb tide did on the English side of the channel; and thus he will carry twelve hours tide with him; whereas had he continued on either side, he would have had a regular six hours tide each way; with this difference, that he would always have the tide setting in opposite directions on the one side, to what it would be on the other. That is; if the flood tide was setting to the *westward* on the English side, the flood tide would, at the same time set to the *eastward* on the Dutch side. Hence were a ship to sail, as above stated, with the wind so that she could lay her course on the *starboard tack*, she ought to get under weigh at *low water* on the English side, by which she would be able to carry twelve hours tide again under her lee, suppposing her to reach the meeting of the tides at high water.

Every person who has been at Spithead may have observed, that the water rises there, and every where *within* the Isle of Wight, as far as Hurst Castle, for more than three hours after it is high water at the Owers, Dunnose, and every where *without* the Wight, and when the ebb tide has, of course, made to the westward; and that it is not high water at Spithead, Portsmouth Harbour, Southampton River, or any where within the Wight, until the ebb tide has run that time.

This is evidently occasioned by the narrow passage between Hurst Castle and the Island not having sufficient capacity to discharge the quantity of water brought by the ebb tide from the eastward through St. Helens; which therefore meeting with a resistance at Hurst Castle, accumulates and *rises* within the Wight, at the same time filling up Portsmouth Harbour, Southampton River, &c. &c. when the tide is falling every where in the English channel.

This circumstance arises from the same cause which occasions the tides to rise and fall in the Straits of Dover; with this difference, that it is high water by the ground, at the last mentioned place when the *flood tide* has run three hours and a quarter from the westward; but it is high water by the ground at the former, when the *ebb tide* has run about the same time from the eastward. It might therefore as well be asserted that the tides meet at St. Helens, Portsmouth Harbour, or Hurst Castle, as at Dungeness; but the fact is, that the phænomena which appear at these different places, are produced by the same cause producing similar effects, with only the difference occasioned by local circumstances in the time and manner; and this cause is the accumulation of the water brought forward by the tide; an accumulation which is occasioned by there not being a sufficient space for its discharge, in consequence of the contraction of the channel at the particular places where these phænomena are exhibited.

There is in fact a meeting of the tides, on a small scale, within the Wight; for the tide of ebb from Southampton River meets the tide of flood from the Needles, at the sand called Bramble (which has probably been originally formed by their meeting); from this they flow to Spithead, and meet

the tide of ebb from Portsmouth Harbour, at the sand called the Spit (perhaps also formed originally by their meeting there), and causing an eddy tide, which would deposit such sand, mud, &c. &c. as the current of the tide brings along with it: nor do I think it at all improbable that the Long Sand, at which I have stated the meeting of the tides through the Downs and from the North Sea to take place, has been likewise formed by the deposit of such things as the opposing tides brought with them, to the place where they met.

Being employed on the expedition against Walcheren, the laborious and difficult duty of passing the transports through the Slough passage into the West Scheldt devolved upon me, and afforded me an opportunity of observing another peculiarity of the tides in that place.

The Slough passage lies between Walcheren and South Beveland, communicating with the West Scheldt and the Veer Gat. From its junction with the last channel, the tide flows through several different channels between the islands, to the northward of South Beveland. On each side of the channels in the Veer Gat and Slough passage are extensive flats or mud banks, which begin to be covered about half tide of flood, and again begin to be dry about half tide of ebb. The flood flows regularly up the West Scheldt, carrying with it a vast body of water, which takes its course by Rammekins through the Slough passage, and meets the flood tide which flows up the Veer Gat at the north end of South Beveland; whence they flow together through the different channels formed by the adjacent islands. At high water the ebb sets again regularly down the West Scheldt and Veer Gat, but the ebb tide in the Slough continues to run to the northward, the same course as the flood tide, and passes down

the Veer Gat until the flats and mud banks become dry; at which time the current of tide in the Slough changes, and runs to the S. Westward into the West Scheldt by Rammekins; thus making the current of tide run nine hours one way, and only three hours the other. This may be accounted for in the following manner: when it is high water in the Scheldt, and the tide of ebb sets down the river, it sets over the extensive flat between the north-west point of South Beveland and Rammekins into the Slough, until the flat becomes dry, which occasions the tide to continue the same course as before, although the water is falling. But when all the flats become dry, and the water is confined within the proper limits of the respective channels of the West Scheldt, the Veer Gat, and the Slough; and the Veer Gat being then only about 30 yards or less in width, three or four times narrower than the Slough, the water through the Slough cannot any longer find a vent through the Veer Gat, and therefore seeks one by the West Scheldt, where there is sufficient space for it; and hence the tide in the Slough changes and runs out by Rammekins into the Scheldt.

It would scarcely have been important to mention this peculiarity, as it is confined to a very small space, and where vessels of any considerable burthen never perhaps passed before the above mentioned expedition, and never may again; had it not on this occasion presented one of the greatest obstacles,* next to the continual adverse gales, which the transports had to contend with, in getting into the West Scheldt; and which could not have been overcome, but by dint of the most laborious and persevering exertions; and also as it furnishes a

* This obstacle, I confidently believe, was never known to the Commander in Chief, down to this moment.

proof of what I have before advanced, "that local circumstances will always have an effect upon the tides, to which no general reasoning can apply, in all straits and insular situations." These circumstances, however, may readily be ascertained by observation and by observation *only*.

J. ANDERSON. CAPT. R.N.

36 Hans Place, 6th February, 1819.

I have annexed a table showing the gradual rising and falling of the tides in Boulogne Bay, from soundings* taken every half hour whilst laying at anchor there, and which I think will greatly tend to confirm what I have advanced, with respect both to the rise and setting of the tides in the Straits of Dover, with the times of high and low water, and of the change of the current of tide there; circumstances which, I have reason to fear, have not been hitherto sufficiently attended to; but which would prove of the utmost importance, especially on expeditions where much boat service must be had recourse to; and in disembarking troops at a particular point, or in making an attack upon vessels at anchor during the darkness of the night; when a want of the necessary knowledge of the tides, or as it has often been called "a mistake in reckoning them," might be productive of the most fatal consequences.†

* As these soundings were taken with a common lead and line, and by different hands, I cannot venture to say that they were taken very accurately; and there might also be some irregularities in the ground, which would occasion a difference; and besides, they were taken 6 or 7 miles from the shore, where the tides do not rise quite so high as on the shore, owing to the re-action of the ground.

† A mistake in calculating the tide at this very place is mentioned by Lord NELSON, as a reason why the boats sent in by him to attack the French flotilla in Boulogne Bay, in 1801, did not get up with the enemy till long after the appointed time.

REFERENCES TO THE TABLE.

The first column contains the month, and day of the month.

The second column contains the wind and weather.

The third column contains the time the soundings were taken:

And the fourth contains the soundings, and the time the ship *tended*, or turned round with the tide. T. E. signifies tended to the eastward or to the flood. T. W. signifies tended to the westward or to the ebb tide.

Soundings taken in Boulogne Bay, at anchor, on 31st July, 1st, 2d, 3d and 4th August, 1811, on Board His Majesty's Sloop Rinaldo, Capt. ANDERSON.

Month and Days.	Wind and Weather.	Time. H.	Time. M.	Soundings in fathoms.
July 31, 1811. P.M.	Fresh Breezes. NE.b N.	Tended to Ed.		
		3	30	13½
		4	—	14
		4	30	14½
		5	—	15½
		5	30	16½
		6	—	16½
		6	30	17
		7	—	H.W. 17½
		7	30	17
		8	—	17
		8	30	16½
		9	—	16½
		9	30	Wd. T. 16
		10	—	15½
		10	30	15½
		11	—	15½
		11	30	15
		Midnight		14½
August 1, 1811. A.M.	Fresh Breezes. NE.bN.	A.M.	30	14½
		1	—	14
		1	30	L.W. 13½
		2	—	14
		2	30	14½
		3	—	14½
		3	30	14½
		4	—	15
		4	30	Ed. T. 15½
		5	—	15½
		5	30	16½
		6	—	16½
		6	30	17½
		7	—	H.W. 17½
		7	30	17½
		8	—	17½
		8	30	17½
		9	—	17½
		9	30	17½
		10	—	17
		10	30	Wd. T. 16½
		11	—	15½
		11	30	15½
		Noon		14½
		P.M.	30	14½
		1	—	14½
		1	30	14½
		2	—	L.W. 14½
		2	30	14½
		3	—	14½
		3	30	14½
		4	—	14½
		4	30	14½
		5	—	T. Ed. 15½
		5	30	15½
		6	—	15½
		6	30	16½
		7	—	17
		7	30	17½

Month and Days.	Wind and Weather.	Time. H.	Time. M.	Soundings in fathoms.
August 2, 1811.	Calm.	8	—	17½
		8	30	17½
		9	—	17½
		9	30	H.W. 17½
		10	—	17½
		10	30	17
		11	—	16½
		11	30	16
		Midnight		15½
		A.M.	30	T. Wd. 14½
		1	—	14½
		1	30	14
		2	—	14
		2	30	14
		3	—	L.W. 14
		3	30	14
		4	—	14
		4	30	14½
		5	—	14½
		5	30	15
		6	—	15½
		6	30	T. Ed. 16
		7	—	16½
		7	30	17
		8	—	17½
		8	30	17½
		9	—	17½
		9	30	18
		10	—	H.W. 18
		10	30	18
		11	—	17½
		11	30	17½
		Noon		16½
		P.M.	30	16
		1	—	T. Wd. 15½
		1	30	14½
		2	—	4 ½
		2	30	14
		3	—	14
		3	30	L.W. 14
		4	—	14
		4	30	14
		5	—	14½
		5	30	15
		6	—	T. Ed. 15½
		7	—	16½
		7	30	17
		8	—	17½
		8	30	17½
		9	—	17½
		9	30	17½
		10	—	18
		10	30	H.W. 18
		11	—	18
		11	30	17½
		Midnight		16½

Month and Days.	Wind and Weather.	Time. H.	Time. M.	Soundings in fathoms.
August 3, 1811.	Light variable airs inclining to calm.	A.M.	30	16½
		1	—	T. Wd. 16
		1	30	15½
		2	—	15½
		2	30	15
		3	—	14½
		3	30	14½
		4	—	L.W. 14½
		4	30	14½
		5	—	14½
		5	30	14½
		6	—	15
		6	30	15½
		7	—	16½
		7	30	T. Ed. 17
		8	—	17½
		8	30	17½
		9	—	18
		9	30	18
		10	—	H.W. 18
		10	30	18
		11	—	18
		11	30	17½
		Noon		17½
		P.M.	30	17
		1	—	16½
		1	30	15½
		2	—	T. Wd. 15
		2	30	14½
		3	—	14½
		3	30	14½
		4	—	14
		4	30	14
		5	—	L.W. 14
		5	30	14
		6	—	14½
		6	30	15
		7	—	15½
		7	30	15½
		8	—	T. Ed. 16
		8	30	16½
		9	—	16½
		9	30	17
		10	—	17½
		10	30	H.W. 17½
		11	—	17½
		11	30	17
		Midnight		17
August 4. 1811.	Moderate Breezes, W.	A.M.	30	17
		1	—	17½
		1	30	17
		2	—	16½
		2	30	15½
		3	—	T. Wd. 15
		3	30	14½
		4	—	14½
		Weighed.		

Full moon, and the tides not yet at their highest.

XVI. *On the Ova of the different tribes of Opossum and Ornithorhynchus. By Sir* Everard Home, *Bart. V. P. R. S.*

Read March 25, 1819.

Now it is determined that the ova of quadrupeds in general are formed in corpora lutea, and that in all such animals the ova become attached to the uterus, and by this means the fœtus receives its support and increase; we are enabled to ascertain the modes of formation of the ova of the opossum tribes, which from the want of this previous knowledge have not been investigated with the smallest degree of success. This becomes the best apology that can be made for the failure of every former attempt.

In this enquiry it will be found, that the ova of all the animals of these tribes are not formed in the same manner, and that the differences met with, make two distinct links between quadrupeds in general and the ornithorhynchi; these again approach so nearly to the bird, as to complete the links of gradation between the human species and the feathered race, so far at least, as concerns their mode of generation.

The mode of formation of the ova in the kangaroo, constitutes the first link in this beautiful series.

In the kangaroo, Mr. Bauer has found the corpus luteum, similar to that in quadrupeds; it is represented in the annexed drawing. Indeed it is to Mr. Bauer's talents and microscopical observations that we are indebted for all our information upon this subject.

In the kangaroo, the ovum when expelled from the corpus luteum, passes along the Fallopian tube, and as there is a thickening and glandular structure surrounding the portion of the tube next to its termination in the uterus, it is there that the yelk, or something analogous to it, is probably secreted; the ovum with the newly acquired yelk, drops from a pendulous opening into the uterus, in which it receives the albumen. In one specimen the uterus in a pregnant state, came under my observation; but as it was sent from New South Wales, in spirit which had not been timely renewed, the contents of the uterus were reduced to a confused mass, in which only a part of the bones could be made out; enough was however seen, to determine that the ovum of the kangaroo in the uterus has an abundant supply of albumen. There was no attachment whatever between the albumen and the uterus. There are two lateral canals that communicate between the uterus and vagina: these answer the purpose of aerating the fœtus by means of atmospheric air.

As the penis of the male has only one orifice at the point adapted to the os tincæ, the ovum must of necessity be impregnated in the uterus, the structure of the Fallopian tubes, and their mode of terminating in the uterus, rendering it impossible for the semen to pass into them.

The fœtus, as soon as it arrives at a certain size (at which time it in general weighs about 12 grains,) is expelled from the uterus, and is received into the marsupium, where it becomes attached to the point of one of the nipples, at first by simple contact; but as the fœtus grows, the nipple is found farther in the mouth upon the surface of the tongue. In the 85th volume of the Philosophical Transactions, Plate XVIII. XIX. XX. and XXI. and in the 100th volume, Plate XIII. most of

these circumstances are represented, but the uses of the different parts, I readily confess, I was at that time unable accurately to comprehend.

The mode of formation of the ova in the Koli and the Wombat of New South Wales, and in the great and small Opossum of North America, constitutes the second link in this chain of gradation.

In none of these different genera are there corpora lutea in the ovaria, but in their place a certain number of yelk-bags of different sizes; and these are so completely imbedded in the substance of the ovarium, that to common observation they appear to be so many corpora lutea. There is no thickened glandular structure surrounding the Fallopian tubes near their termination in the uterus. Instead of one uterus having two Fallopian tubes, there are two uteri and one tube to each; and in proof that the ovum in each uterus is impregnated separately in its own cavity, the point of the penis in the male is so formed as to throw the semen into both. The lateral tubes, by which the fœtus is aerated in the kangaroo, are formed in these genera in a different manner; there is only one to each uterus, and this, instead of communicating with the uterus at the fundus, opens into it at the cervix. The yelk-bags are shown in the annexed drawing. I have not had an opportunity of examining the ovum of any of these animals in utero, but Mr. BELL, a very intelligent surgeon, transmitted an account to Sir JOSEPH BANKS, of the dissection of a female koli, in which he met with an imperfectly formed embryo in each of the uteri, surrounded by a mass of albumen. The young of all these genera are expelled from the uterus into the marsupium, and become attached to the prominent points of the nipples.

The mode of formation of the ova in the ornithorhynchi, constitutes the intermediate link between that of the American opossum and the bird.

The yelk-bags in the ovaria of the ornithorhynchi are more distinct, and less deeply imbedded than in the opossum: there is no regular uterus, nor Fallopian tubes; the yelk-bags pass along an oviduct, the lower part of which performs the office of a uterus. In this situation the ova are impregnated; the penis of the male, which is bifid, throwing the semen into both oviducts at the same time, through several points like a watering pot, so as to scatter it all over the cavity. The ova are aerated by the vagina. The ova in a magnified state are represented in the annexed drawing. The organs of generation are figured in the 92nd volume, Plate IV. of the Philosophical Transactions. To show that the yelk-bags in the ornithorhynchus resemble those of the pullet, a magnified drawing of them made by Mr. Bauer in that bird, is annexed.

To those members not conversant in comparative anatomy, the following summary may be acceptable.

In the human species, and quadrupeds in general, the ova are formed in corpora lutea, and pass into the uterus, to the sides of which they become attached; when the fœtus is completely formed it is expelled by the vagina, and afterwards sucks the mother.

In the kangaroo the ova are formed in corpora lutea, receive their yelks in the Fallopian tube, and their albumen in the uterus. The ovum thus completed, is impregnated in the uterus, aerated by means of lateral tubes, and when the young is expelled from the uterus, it is received into the marsupium, and attached to the nipple of the mother.

In the American opossum, the yelk bags are formed in the ovaria; pass into the uteri, there receive the albumen, and are then impregnated; the fœtus in each uterus is aerated by one lateral tube. When expelled from these uteri, the young are received into the marsupium, and become attached to the nipples of the mother.

In the ornithorhynchi the yelk-bags are formed in the ovaria; received into the oviducts, in which they acquire the albumen, and are impregnated afterwards; the fœtus is aerated by the vagina, and hatched in the oviduct, after which the young provides for itself, the mother not giving suck.

In the pullet, the yelk-bags are formed in one ovarium, impregnated in one oviduct, and hatched out of the body.

EXPLANATION OF THE PLATES.

PLATE XVI.

Ovarium of the Kangaroo.

Fig. 1. The ovarium of a young kangaroo laid open: natural size; showing the corpus luteum.

Fig. 2. The same section magnified four diameters, to show the corpus luteum more distinctly.

Fig. 3. A section of the corresponding ovarium of the same kangaroo, magnified four diameters; to show an incipient corpus luteum.

Fig. 4. A similar section of the ovarium of an old kangaroo: natural size; in which there are two corpora lutea.

Fig. 5. The same section, magnified four diameters.

Plate XVII.

Ovarium of the American Opossum.

Fig. 1. Front view of the ovarium of the large American opossum: natural size.

Fig. 2. The same view, magnified five diameters.

Fig. 3. Back view, magnified five diameters.

Fig. 4. A perpendicular section, magnified five diameters.

Fig. 5. The same section, magnified ten diameters.

Fig. 6. A young yelk-bag, magnified twenty diameters.

Fig. 7. A full grown yelk-bag, magnified twenty diameters.

Fig. 8. A full grown yelk-bag opened, to show its contents; magnified twenty diameters.

Plate XVIII.

Ovarium of the Ornithorhynchus Paradoxus.

Fig. 1. Front view of the ovarium of the ornithorhynchus paradoxus: natural size.

Fig. 2. The same view, magnified five diameters.

Fig. 3. Back view, magnified five diameters.

Fig. 4. A small portion cut off from the upper end of the left side of Fig. 3.; magnified ten diameters.

Fig. 5. An internal view of the same portion, magnified ten diameters, to show the yelk-bags.

Fig. 6. A full grown yelk-bag, magnified twenty diameters.

Fig. 7. A young yelk-bag opened, to show its contents: magnified twenty diameters.

Fig. 8. A full grown yelk-bag opened, to show its contents: magnified twenty diameters.

Fig. 9. The globules of the yelk diluted with water, magnified four hundred diameters.

PLATE XIX.

Ovarium of the Hen.

Fig. 1. Front view of the ovarium of a hen: natural size.

Fig. 2. A small portion of the same, with some very young yelks: natural size.

Fig. 3. The same small portion, magnified five diameters.

Fig. 4. Back view of the ovarium of the hen: natural size.

Since this Paper was sent to the press, the author has received, through the kindness of Governor MAC QUARRIE and Sir JOHN JEMISON, four specimens of female Ornithorhynchi Paradoxi from New South Wales, and finds in all of them, as well as in every other specimen that has come under his observation, that there are yelk-bags only in the left ovarium, showing that both ovaria are not generally in use at the same time. This is an approach to the bird, in which there is only one, lying on the left side.

In the chick of the common fowl before it is hatched, there is a small portion of an oviduct on the right side, but his disappears before the chick is completely formed.

Fig 1

2

3

4

5

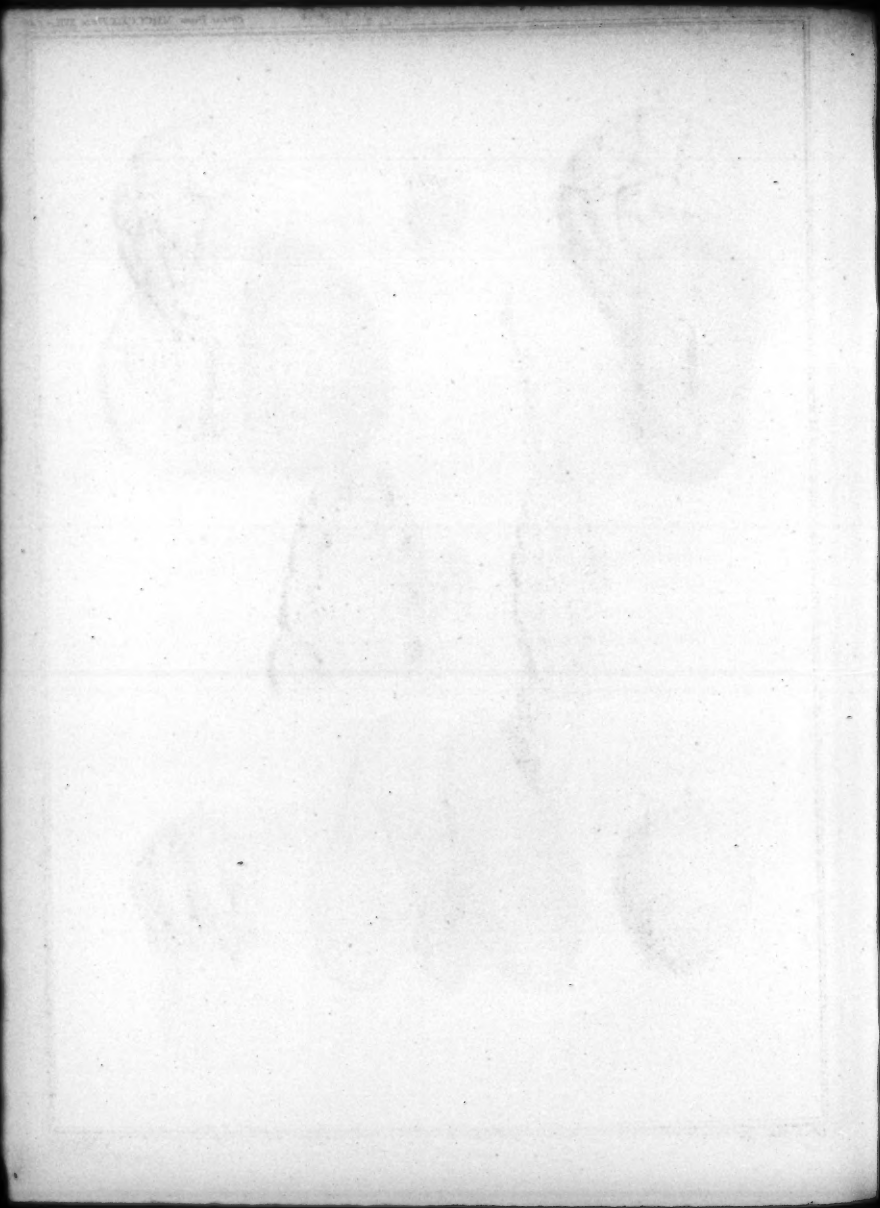
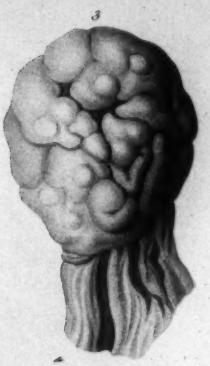
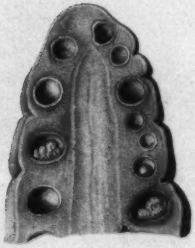


Fig. 1.





Drawn Chaves, del.

W. Bass sc.

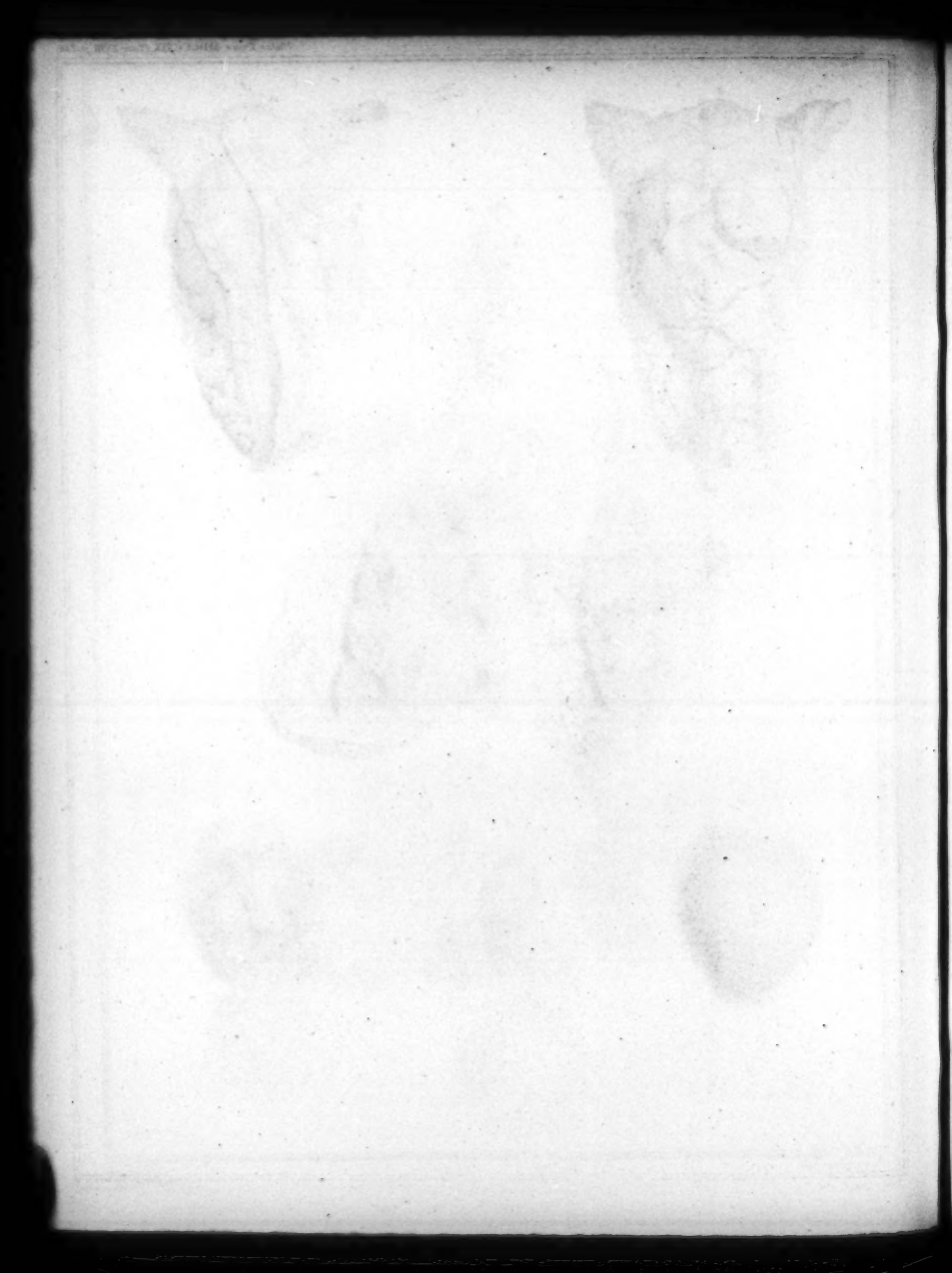
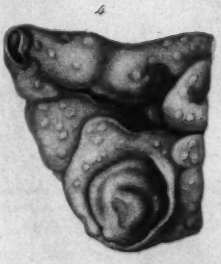
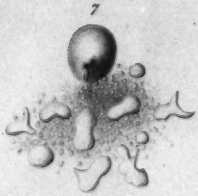
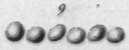
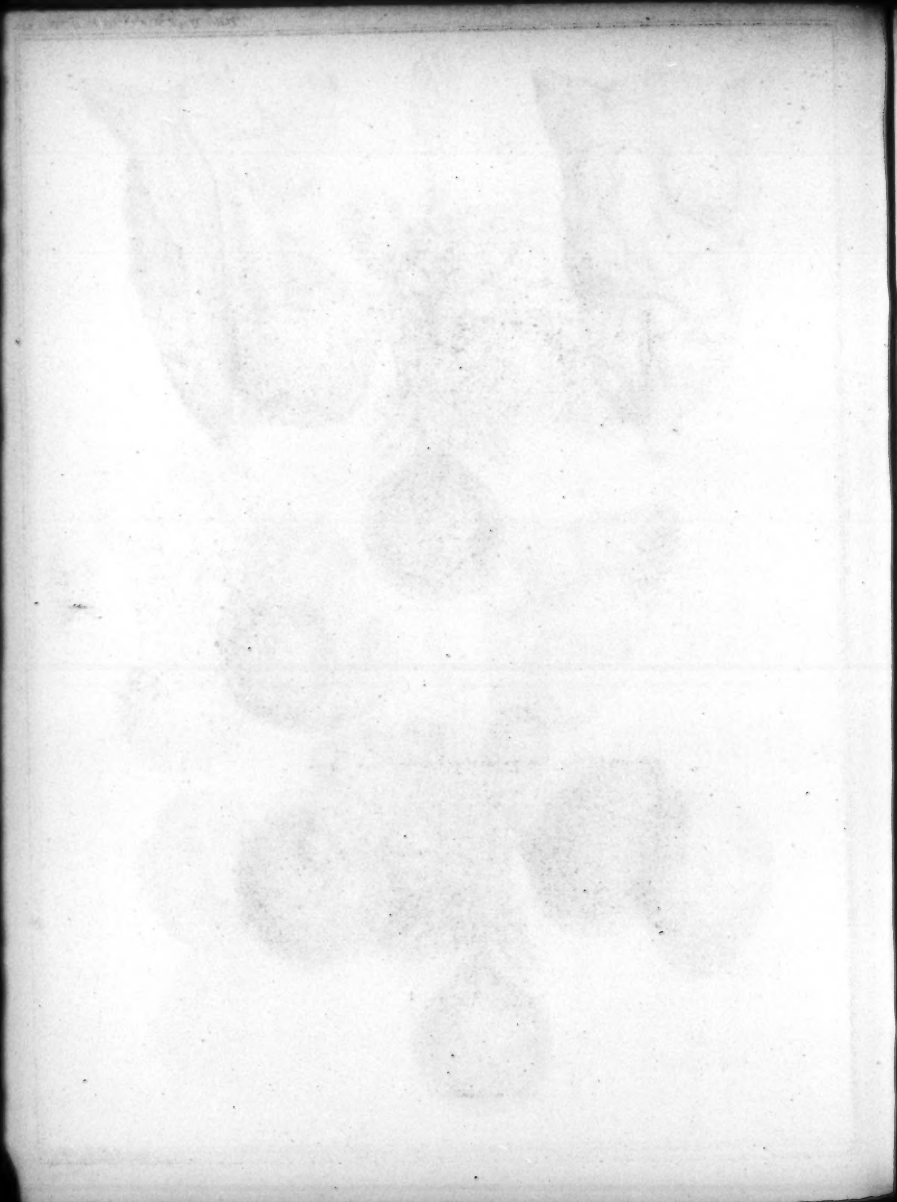


Fig. 1.









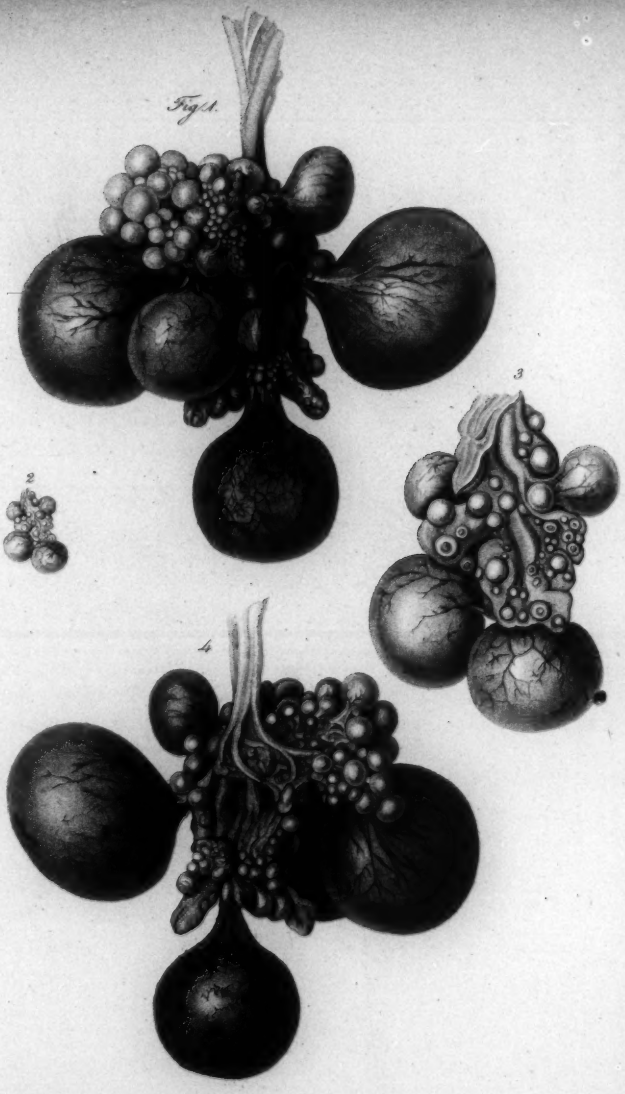


Fig 1.
2
3
4

XVII. *The results of Observations made at the Observatory of Trinity College, Dublin, for determining the Obliquity of the Ecliptic, and the Maximum of the Aberration of Light. By the Rev.* J. Brinkley, *D. D. F. R. S. and M. R. I. A. and Andrews's Professor of Astronomy in the University of Dublin.*

Read April 1, 1819.

OBSERVATIONS have been made by the eight feet circle of the Observatory of Trinity College, Dublin, at the respective summer solstices since the year 1809, with the exception of two. The obliquity of the ecliptic thence resulting, has always agreed so nearly with that adopted in the French tables, that I have heretofore thought it useless to make any public communication relative thereto. But some circumstances have now induced me to lay my results before the Royal Society.

The recent publication of Mr. BESSEL's valuable labours on the observations of Dr. BRADLEY, has afforded us a more exact determination of the obliquity of the ecliptic, as deduced from the early observations by the Greenwich quadrant, than we before possessed. The comparison of this with the present obliquity, gives us the diminution for an interval of nearly 60 years, with a considerable degree of accuracy, and almost sufficient to enable us to state with some confidence the mass of Venus.

To obtain this point with a greater degree of certainty,

the present obliquity, as deduced from a mean of the observations of different astronomers, should be used.

It has been an opinion almost generally received among astronomers, that observations of the winter solstice, have given a less obliquity of the ecliptic than observations of the summer solstice.

The explanation of this seemed very difficult. But in the above mentioned work of Mr. BESSEL, he calls in question this opinion, and shows that the observations of Dr. BRADLEY give the same result, both in summer and winter. His own observations also tend to the same conclusion. The observations of Dr. MASKELYNE, of M. ORIANI, of M. ARAGO, and of Mr. POND, are in opposition to these; to which my own may be added.

It is not likely that this difference really exists, but it is a question of some importance in astronomy, and the explanation thereof may throw some light on other points.

It is probable the difference arises from some unknown modification of refraction. I find, and I believe other observers have found the same, that at the winter solstice, an irregularity of refraction takes place for the sun greater than for the stars, at the same zenith distance. The zenith distance of the sun at this place is then nearly 77°.

What Mr. BESSEL has adduced, certainly tends to render the prevalent opinion doubtful. It therefore appears to me of consequence, that astronomers should pay attention to the observations at the winter solstice. My observations at that time have been much fewer than in the summer, because, on account of the uncertainty of refraction, I considered them of less importance.

It has been proposed to make the two results agree, by an increase of the quantity of BRADLEY's mean refraction; but this could not be done, without increasing it by a quantity greater than can be justified by other determinations respecting refraction.

Considering then this uncertainty respecting the observations of the winter solstice, it appears better to compare the results from Dr. BRADLEY's summer solstices, with the result as deduced from the mean of the observations of different astronomers.

Mean Obliquity, Jan. 1, 1813.

M. ORIANI*	4 summer solstices	23°	27′	50″,34
Mr. POND†	2 summer solstices	23	27	50 ,37
Mr. ARAGO‡	2 summer solstices	23	27	50 ,09
Dr. BRINKLEY	8 summer solstices	23	27	50 ,99
Mean Jan. 1. 1813		23	27	50 ,45
Dr. BRADLEY, Jan. 1. 1755		23	28	15 ,49
diff.	58 years.			25 ,04

This gives 0,″ 43, for the annual diminution.

The mean of 18 observations near the winter solstice gives me mean obliquity Jan. 1, 1813, 23° 27′ 48″,14.

The above determination of the obliquity by observations near the summer solstice gives (taking the annual diminution 0″,43.)

Mean obliquity Jan. 1, 1800=23° 27′ 56″,0, differing only 1″ from that assumed in M. DELAMBRE's tables of the sun.

* See Mr. BESSEL's work, p. 62.

† Phil. Trans. 1813, p. 304. This is corrected for the solar nutation.

‡ Conn. des Temps. 1816. The observations were made with a three feet repeating circle.

And as far as my own observations are concerned, the difference does not exceed half a second.

In M. ZACH's solar tables, there is given a determination of the obliquity of the ecliptic computed by M. GERSTNER, from a mean of a great many observations of Dr. MASKELYNE's, made at 19 summer solstices. Although the results of the several solstices are rather discordant, more so than was to be expected from a fixed instrument, yet it is likely a mean of 178 observations cannot be far from the truth.

This mean is 23° 28′ 11″,0 for 1769,

when reduced to 1800, is 23 27 57 ,7,

which agrees sufficiently near with the present determination, to show that, if the necessary corrections for the sun's latitude, &c. had been used, the result would probably have been very exact.

The mean of 102 observations at 17 winter solstices computed by M. GERSTNER, gives for 1769—23° 28′ 8″; a result which, after making all possible allowances for the error of the quadrant, is considerably less than that deduced from the summer solstices.

In using the eight feet circle, two or more observations were made a few minutes before the sun arrived at the meridian, and then the instrument was reversed, and observations made after the passage. The results were carefully reduced to the meridian; the upper and lower limbs being observed, the zenith distance of the centre was deduced from the instrument itself. This facility of reversing the instrument seems more likely to produce exact results, than those obtained by a fixed instrument, although from the necessary effect of the action of the sun on the parts of the instrument, the results must be expected to be more discordant than those obtained by a fixed instrument.

The results of the several observations are as follow.

Time of Observation.	Observed Declination.	Corr. for ☉'s Lat.	Mean Obliquity reduced to Jan. 1, 1812.
	° ′ ″	″	° ′ ″
1809. June 9	23 56 4,34	+ 0,24	23 27 51,43
14	23 16 24,84	+ 0,82	50,85
15	23 19 15,49	+ 0,85	50,76
17	23 23 41,83	+ 0,84	49,56
18	23 25 15,58	+ 0,82	46,67
19	23 26 28,74	+ 0,74	47,87
22	23 27 37,58	+ 0,32	49,58
27	23 21 17,24	— 0,40	52,76
1810. June 1	22 0 37,31	+ 0,49	23 27 50,00
6	22 37 23,04	+ 0,64	47,55
20	23 27 7,65	— 0,55	49,43
22	23 27 43,28	— 0,29	53,43
1811. June 18	23 24 35,59	+ 0,63	23 27 52,07
19	23 25 58,05	+ 0,68	51,07
22	23 27 40,65	+ 0,66	50,78
1813. June 22	23 27 41,28	+ 0,25	23 27 53,58
24	23 26 17,22	+ 0,34	50,07
25	23 24 59,75	+ 0,33	49,97
26	23 23 17,68	+ 0,28	50,06
28	23 18 38,88	+ 0,05	49,36
1814. June 15	23 18 40,32	+ 0,65	23 27 49,01
19	23 26 21,29	+ 0,07	51,22
21	23 27 40,26	— 0,26	49,63
22	23 27 42,88	— 0,42	49,23
23	23 27 21,44	— 0,56	49,63
24	23 26 35,25	— 0,65	50,02
25	23 25 23,89	— 0,69	50,12
1815. June 21	23 27 41,40	+ 0,03	23 27 52,78
22	23 27 48,76	+ 0,10	51,48
27	23 22 24,08	+ 0,73	54,56
28	23 20 1,40	+ 0,76	51,46
29	23 17 16,13	+ 0,76	50,45
1816. June 16	23 22 29,42	+ 1,15	23 27 52,31
21	23 27 50,73	+ 0,58	51,23
28	23 18 3,91	— 0,11	53,61
1818. June 11	23 4 50,08	— 0,76	23 27 49,23
12	23 9 2,50	— 0,64	53,29
18	23 25 20,22	+ 0,33	54,81
20	23 27 26,32	+ 0,07	53,23
22	23 27 55,72	+ 0,51	53,53
24	23 26 44,04	+ 0,35	51,92
30	23 13 20,25	— 0,42	51,53

In the paper which I had the honour of presenting to the Royal Society last year, I mentioned my doubts as to the quantity of the maximum of the aberration of light, and that, as far as could be ascertained from Dr. BRADLEY's Wanstead observations with a zenith sector, we ought rather to adopt 20″,00 than 20″,25. I also mentioned that it would be desirable to investigate this point, and therefore during the last year, I instituted a course of observations for this purpose, and I beg leave to offer the results thereof.

	No. Ob.	Max. Aber.	N. P. D. By Observations in 1818.	N.P.D. Before.
		″	° ′ ″	″
α Cassiopeæ	22	20,72	34 27 43,54	43,59
Polaris	23	20,73	1 39 44,55	44,27
α Urs. Maj.	23	20,04	27 16 7,50	7,38
γ	27	21,20	35 17 34,83	36,22
ι	30	21,36	33 3 0,26	0,45
ζ	20	20,15	34 7 15,31	17,03
η	21	21,12	39 46 29,15	29 37
	166	20,80		

By these the maximum appears to be 20″,80, which is much greater than I had expected. While these observations were going forward, Mr. BESSEL's work above mentioned was published. From several investigations in the Greenwich observations of Dr. BRADLEY, he also deduced the maximum = 20″,70, nearly. These results certainly appear extraordinary, and are not likely to be acknowledged by astronomers, unless they shall be established by a great number of observations.

My results were computed with great care, allowances being made for the ellipticity of the earth's orbit. It is not likely, supposing the velocity of the light of all the stars

to be the same, that the result can err more than $\frac{1}{4}$ of a second.*

By continuing the observations, I hope to obtain farther information on this interesting point. And it appears to be an enquiry deserving of the joint co-operation of astronomers. Those instruments which admit of observing each star, without a reference to other stars, seem best adapted thereto. It is not likely that the maximum of aberration differs in different stars; yet this ought not to be taken for granted.

The mean N.P.D. Jan. 1, 1818, deduced from former observations, have been put down as a proof of the consistency of my instrument. ζ Ursæ Majoris is the only star in which the difference is worth notice. Whether this difference is from the error of observation, or from any uncertainty in the proper motion of the star, it is difficult to say. Three results reduced by BRADLEY's refraction are as follow.

		N.P.D. Jan. 1, 1815.
My observation,	1812	$34^{\circ}\ 6'\ 19''.99$
Mr. POND's observation,	1815	18 .92
My observation,	1818	17 .67

A comparison of independent results is for many reasons much to be desired. I offer the above principally with a view of calling the attention of astronomers to such investigations.

* The observations of Mr. POND with the fixed telescope, may be adduced as contrary to my results; because with this maximum of aberration, his summer and winter differences of N. P. distance of β Aurigæ and α Cygni would differ by $1''$ in a direction contrary to parallax. But it also seems to show the necessity of exact determination of the precise quantities of the equations for N. P. D. before any conclusive arguments respecting the non-existence of parallax, from observations of the positions of stars relative to each other, can be adduced. In observations by the eight feet circle this is not so necessary, as has been before mentioned.

It appears to me, that the only method by which an explanation of the difficulties that have occurred, from a comparison of the Greenwich observations and of those made at this Observatory, can be obtained, is from an *extensive* series of observations of many stars, referring each to the apparent zenith point. I am therefore pursuing such a course of observations. Conclusions as to the existence or non-existence of parallax, from comparisons of the *relative* places of stars taken indiscriminately, must be liable to much uncertainty, whether the comparisons be made by polar distances or by right ascensions. The former being affected by the uncertainty of refraction, may, at first view, be thought more subject to error than the latter; but a careful consideration of the circumstances attending the latter method, will show that it has its peculiar difficulties.*

* As Mr. BESSEL's determination of the maximum of aberration has been referred to, it may also be right to mention his results respecting the parallax of certain stars. He uses transit observations of stars nearly opposite in right ascension (p. 110. &c.) Thus he finds the sum of the parallaxes of Sirius and α Lyræ insensible, and the sum of the semi-parallaxes of Procyon and α Aquilæ, nearly 1″. This method of using the transit observations is undoubtedly far preferable to that of using them indiscriminately. With respect to the observations Mr. BESSEL had to compute from, I think it must be allowed they were not sufficiently exact, to give much weight to his conclusions. The methods of observing with the transit, and of entering the observations, were then far inferior to the present. This objection, however, does not apply to the observations of the pole star, and therefore does not affect the maximum of aberration deduced from the observed right ascension of that star.

Observatory, Trinity College, Dublin, February 18, 1819.

XVIII. *On some new Methods of investigating the Sums of several Classes of infinite Series.* *By* Charles Babbage, *Esq. A. M. F. R. S.*

Read April 1, 1819.

THE processes which it is the object of this paper to explain, were discovered several years since; but certain difficulties connected with the subject, which I was at that time unable to explain, and which were equally inexplicable to several of my friends, to whom I had communicated these methods, induced me to defer publishing them, until I could offer some satisfactory solution.

These observations refer more particularly to the second method which I have detailed in this paper, and which may not inappropriately be called the *method of expanding horizontally and summing vertically*. Some traces of this method may, perhaps, be found in former writers, and particularly in a paper by Professor Vince, "On the Summation of Series," printed in the Philosophical Transactions for 1791; but there exists this peculiarity in that which I have employed, that after a certain number of the vertical columns are summed, all the remainder either vanish, or else have some common factor. This method, which I employed about the year 1812, gave the values of a variety of series whose sums had not hitherto been known, most of which were apparently correct, but some of the consequences which followed were evidently erroneous. About this time, Mr. HERSCHEL, to whom I had

communicated these anomalous results, by following a very different course, arrived at several general theorems, which, when applied to the series I had obtained, gave the same results. This coincidence at first increased my confidence in the values so discovered, and I continued to examine the reason why my own formulæ were in some cases defective. Mr. HERSCHEL's method was published in the Philosophical Transactions for 1814; and it was not until some time after that I perceived, that although the investigations were very different, the fundamental principle was the same in both methods. This induced me to attempt summing the same series by a direct process, and I succeeded in obtaining their sums by integration relative to finite differences, aided by certain peculiar artifices. The results obtained by this new plan, which is the first treated of in this paper, coincided with those already found, and seemed to confirm their truth, without in the least indicating the cause of the error: this cause however I now began to suspect, and, after some enquiry, I was at length able to detect. I have found that the *method of expanding horizontally and summing vertically*, will always lead to correct results, provided a certain series which I have pointed out, is finite. I have also shown how to express this series by a definite integral; and when this integral or this series has a finite value, the method may be depended on. In case this series or this definite integral is not finite, then the value of the series* multiplied by zero, must be added to

* The investigation of this series is generally a task of considerable difficulty. I have however given an example, wherein the correction thus found, added to the sum indicated by the method we are considering, gives the true value of the series, which in this case is one whose sum has been found by Euler.

the sum given by this method. In this latter case, however, the mode of summation which I have proposed, is not well adapted for giving the sums of series; its greatest advantage is felt when the integral or series alluded to is finite: but even in this case the criterion I have pointed out is not useless, for it serves to except certain particular values of the variables, which would give incorrect results. Without this criterion, or without something equivalent to it, I am inclined to think that the principle on which this method is founded, although it will probably in many cases give accurate results, will in others produce such as are not only numerically but symbolically untrue. It is worthy of remark, that the *method of expanding horizontally and summing vertically*, in many instances, gives precisely the same formulæ as the direct process of integration; yet that that method attaches limitations to them, which are necessary to their accuracy, but which are not indicated by the method last mentioned.

Before I proceed to explain these two processes, it will be convenient to prove that the values of all series of the forms

$$\mathop{\mathrm{A}}_{1}x\frac{(\sin\theta)^n}{(\cos\theta)^n}+\mathop{\mathrm{A}}_{2}x^2\frac{(\sin 2\theta)^n}{(\cos 2\theta)^n}+\mathop{\mathrm{A}}_{3}x^3\frac{(\sin 3\theta)^n}{(\cos 3\theta)^n}+\&c.$$

$$\mathop{\mathrm{A}}_{1}x\frac{(\cos\theta)^n}{(\sin\theta)^n}+\mathop{\mathrm{A}}_{2}x^2\frac{(\cos 2\theta)^n}{(\sin 2\theta)^n}+\mathop{\mathrm{A}}_{3}x^3\frac{(\cos 3\theta)^n}{(\sin 3\theta)^n}+\&c.$$

depend on series of the form

$$\mathop{\mathrm{A}}_{1}\frac{x}{(\cos\theta)^n}+\mathop{\mathrm{A}}_{2}\frac{x^2}{(\cos 2\theta)^n}+\&c.$$

$$\mathop{\mathrm{A}}_{1}x\frac{\sin\theta}{(\cos\theta)^n}+\mathop{\mathrm{A}}_{2}x^2\frac{\sin 2\theta}{(\cos 2\theta)^n}+\&c.$$

and

$$\mathop{\mathrm{A}}_{1}\frac{x}{(\sin\theta)^n}+\mathop{\mathrm{A}}_{2}\frac{x^2}{(\sin 2\theta)^n}+\&c.$$

$$\mathop{\mathrm{A}}_{1}x\frac{\cos\theta}{(\sin\theta)^n}+\mathop{\mathrm{A}}_{2}x^2\frac{\cos 2\theta}{(\sin 2\theta)^n}+\&c.$$

or else they depend partly on these and partly on other series, containing the powers of the sines or cosines of an arc in arithmetical progression in their numerators, which is a species whose sums are easily found. For the sake of brevity, I shall make use of the general term of any series with the characteristic S prefixed to it to denote that series. Beginning then with the series $\underset{i}{\mathrm{S}}Ax^i\frac{(\sin i\theta)^m}{(\cos i\theta)^n}$, we observe that when m is an even number, we have

$$\underset{i}{\mathrm{S}}Ax^i\frac{(\sin i\theta)^m}{(\cos i\theta)^n}=\underset{i}{\mathrm{S}}Ax^i\frac{\left\{1-(\cos i\theta)^2\right\}^{\frac{m}{2}}}{(\cos i\theta)^n}=\underset{i}{\mathrm{S}}Ax^i\frac{1}{(\cos i\theta)^n}-\frac{m}{2}\underset{i}{\mathrm{S}}Ax^i\frac{1}{(\cos i\theta)^{n-2}}$$
$$+\frac{m.m-2}{2.4}\underset{i}{\mathrm{S}}Ax^i\frac{1}{(\cos i\theta)^{n-4}}-\&c. \qquad (a)$$

this series will always terminate when m is an even number; and if m is greater than n, the last term will have no cosines in its denominator: if $m=n$, the last term will be $\underset{i}{\mathrm{S}}Ax^i$; and if m is less than n, the last term will be $\underset{i}{\mathrm{S}}Ax^i\frac{1}{(\cos i\theta)^{n-m}}$; so that in all cases when m is an even number, the series in question will depend on series of the form $\underset{i}{\mathrm{S}}Ax^i\frac{1}{(\cos i\theta)^n}$, or on others whose sums are known.

Let us now consider the case of $m=$ an odd number; then we have

$$\underset{i}{\mathrm{S}}Ax^i\frac{(\sin i\theta)^m}{(\cos i\theta)^n}=\underset{i}{\mathrm{S}}Ax^i\frac{\sin i\theta.\left\{1-(\cos i\theta)^2\right\}^{\frac{m-1}{2}}}{(\cos i\theta)^n}=$$

$$=\underset{i}{\mathrm{S}}Ax^i\frac{\sin i\theta}{(\cos i\theta)^n}-\frac{m-1}{2}\underset{i}{\mathrm{S}}Ax^i\frac{\sin i\theta}{(\cos i\theta)^{n-2}}+\frac{m-1.m-3}{2.4}-\underset{i}{\mathrm{S}}Ax^i\frac{\sin i\theta}{(\cos i\theta)^{n-4}}+\&c. \qquad (b)$$

This series always terminates when m is an odd number; and in a similar manner we shall find the two following:

$$\underset{i}{\mathrm{SA}}x^i\frac{(\cos i\theta)^m}{(\sin i\theta)^n}=\underset{i}{\mathrm{SA}}x^i\frac{\left\{1-(\sin i\theta)^2\right\}^{\frac{m}{2}}}{(\sin i\theta)^n}=$$

$$=\underset{i}{\mathrm{SA}}x^i\frac{1}{(\sin i\theta)^n}-\frac{m}{2}\underset{i}{\mathrm{SA}}x^i\frac{1}{(\sin i\theta)^{n-2}}+\frac{m\cdot m-2}{2\cdot 4}\underset{i}{\mathrm{SA}}x^i\frac{1}{(\sin i\theta)^{n-4}}-\&c.\ (c)$$

when m is an even number, and

$$\underset{i}{\mathrm{SA}}x^i\frac{(\cos i\theta)^m}{(\sin i\theta)^n}=\underset{i}{\mathrm{SA}}x^i\frac{\cos i\theta\left\{1-(\sin i\theta)^2\right\}^{\frac{m-1}{2}}}{(\sin i\theta)^n}=$$

$$=\underset{i}{\mathrm{SA}}x^i\frac{\cos i\theta}{(\sin i\theta)^n}-\frac{m-1}{2}\underset{i}{\mathrm{SA}}x^i\frac{\cos i\theta}{(\sin i\theta)^{n-2}}+\frac{m-1\cdot m-3}{2\cdot 4}\underset{i}{\mathrm{SA}}x^i\frac{\cos i\theta}{(\sin i\theta)^{n-4}}\&c.\ (d)$$

when m is an odd number.

Let us now propose to investigate the sum of the series

$$\frac{\underset{1}{\mathrm{A}}x}{(\sin\theta)^n}+\frac{\underset{2}{\mathrm{A}}x^2}{(\sin 2\theta)^n}+\frac{\underset{3}{\mathrm{A}}x^3}{(\sin 3\theta)^n}\ \&c.$$

Assume $$\psi x=\underset{1}{\mathrm{A}}x+\underset{2}{\mathrm{A}}x^2+\underset{3}{\mathrm{A}}x^3+\&c.$$

Put $v^{2\alpha}$ for x; then it becomes

$$\psi v^{2\alpha}=\underset{1}{\mathrm{A}}v^{2\alpha}+\underset{2}{\mathrm{A}}v^{4\alpha}+\underset{3}{\mathrm{A}}v^{6\alpha}+\&c.$$

Integrate both sides, observing that $\Sigma v^{2i\alpha}=\frac{v^{2\alpha}}{v^{2\alpha}-1}$; then we have

$$\Sigma\psi v^{2\alpha}=\underset{1}{\mathrm{A}}\frac{v^{2\alpha}}{v^{2\alpha}-1}+\underset{2}{\mathrm{A}}\frac{v^{4\alpha}}{v^{4\alpha}-1}+\underset{3}{\mathrm{A}}\frac{v^{6\alpha}}{v^{6\alpha}-1}+\&c.$$

Integrate again, and after the n^{th} integration we shall have

$$\Sigma^n\psi v^{2\alpha}=\underset{1}{\mathrm{A}}\frac{v^{2\alpha}}{(v^{2\alpha}-1)^n}+\underset{2}{\mathrm{A}}\frac{v^{4\alpha}}{(v^{4\alpha}-1)^n}+\underset{3}{\mathrm{A}}\frac{v^{6\alpha}}{(v^{6\alpha}-1)^n}+\&c.$$

Now let $v=\cos\theta\pm\sqrt{-1}\sin\theta$; then our equation becomes

$$(2\sqrt{-1})^n\Sigma^n\psi^{2\alpha}=\underset{1}{\mathrm{A}}\frac{v^{2\alpha n}-v^{-2\alpha n}}{(\sin\theta)^n}+\underset{2}{\mathrm{A}}\frac{v^{4\alpha n}-v^{-4\alpha n}}{(\sin 2\theta)^n}+\underset{3}{\mathrm{A}}\frac{v^{6\alpha n}-v^{-6\alpha n}}{(\sin 3\theta)^n}+\&c.$$

Put $z+\frac{\pi}{2}$ for z, and we have

$$(2\sqrt{-1})^n\Sigma^n\psi v^{2z+\pi}=\underset{1}{\mathrm{A}}\frac{v^{2\alpha n}}{(\sin\theta)^n}+\underset{2}{\mathrm{A}}\frac{v^{4\alpha n}}{(\sin 2\theta)^n}+\underset{3}{\mathrm{A}}\frac{v^{6\alpha n}}{(\sin 3\theta)^n}+\&c.$$

$$= A_1 \frac{x}{(\sin\theta)^s} + A_2 \frac{x^2}{(\sin 2\theta)^s} + A_3 \frac{x^3}{(\sin 3\theta)^s} + \&c. \qquad (1)$$

If after the integration we had put $\overline{v}^{-1}$ instead of v, and then $v = \cos\theta \pm \sqrt{-1}\sin\theta$, we should have found

$$(-2\sqrt{-1})^s \Sigma^s \psi \overline{v}^{-2x-n} = A_1 \frac{\overline{v}^{-x}}{(\sin\theta)^s} + A_2 \frac{\overline{v}^{-2x}}{(\sin 2\theta)^s} + A_3 \frac{\overline{v}^{-3x}}{(\sin 3\theta)^s} + \&c.$$

$$= A_1 \frac{\overline{x}^{-1}}{(\sin\theta)^s} + A_2 \frac{\overline{x}^{-2}}{(\sin 2\theta)^s} + A_3 \frac{\overline{x}^{-3}}{(\sin 3\theta)^s} + \qquad (2)$$

Neither of the integrals exhibited in (1) and (2) are integrable in the most simple cases, and it is only by their combination that I have been able to obtain the sums of any series. Let us suppose $A = 1$, $A_2 = -1$, $A_3 = 1$, $A_4 = -1$, &c. then

$\psi v^{2x+n} = \frac{v^{2x+n}}{1+v^{2x+n}}$ and $\psi v^{-2x-n} = \frac{v^{-2x-n}}{1+v^{-2x-n}}$; also let $n = 1$; then the difference of the two series is

$$(2\sqrt{-1})\,\Sigma\left\{\frac{v^{2x+1}}{1+v^{2x+1}} + \frac{v^{-2x-1}}{1+v^{-2x-1}}\right\} = \frac{x-\overline{x}^{-1}}{\sin\theta} - \frac{x^2-\overline{x}^{-2}}{\sin 2\theta} + \frac{x^3-\overline{x}^{-3}}{\sin 3\theta}\,\&c.$$

but the integral on the left side of this equation becomes $(2\sqrt{-1})\,\Sigma(1)$ which is equal to $2\sqrt{-1}\,(x+b)$: hence, since $x = \frac{\log x}{2\log v}$ we have

$$2\sqrt{-1}\left\{\frac{\log x}{2\log v} + b\right\} = \frac{x-\overline{x}^{-1}}{\sin\theta} - \frac{x^2-\overline{x}^{-2}}{\sin 2\theta} + \frac{x^3-\overline{x}^{-3}}{\sin 3\theta} + \&c.$$

If $x = 1$, $b = 0$, and since $\log v = \theta\sqrt{-1}$, our series becomes

$$\frac{\log x}{\theta} = \frac{x-\overline{x}^{-1}}{\sin\theta} - \frac{x^2-\overline{x}^{-2}}{\sin 2\theta} + \frac{x^3-\overline{x}^{-3}}{\sin 3\theta} - \&c. \qquad (3)$$

let $x = \cos\theta' \pm \sqrt{-1}\sin\theta'$; then, since $\log x$ will become $\theta'\sqrt{-1}$, by dividing both sides by $2\sqrt{-1}$, we shall have

$$\frac{\theta'}{\theta} = \frac{\sin\theta'}{\sin\theta} - \frac{\sin 2\theta'}{\sin 2\theta} + \frac{\sin 3\theta'}{\sin 3\theta} - \&c. \qquad (4)$$

The series (3) is integrable when multiplied by $\frac{dx}{x}$, and this operation may be repeated any number of times; the first

operation produces

$$\frac{(\log x)^2}{2\theta} \pm \underset{1}{c} = \frac{x+\overset{-1}{x}}{1 \sin \theta} - \frac{x^2+\overset{-2}{x}}{2 \sin 2\theta} - \frac{x^3+\overset{-3}{x}}{3 \sin 3\theta} + \&c.$$

the value of the constant c, which is equal to the series

$$\underset{1}{c} = 2 \left\{ \frac{1}{1 \sin \theta} - \frac{1}{2 \sin 2\theta} + \frac{1}{3 \sin 3\theta} - \&c. \right\}$$

cannot be determined from this equation, but by a second multiplication by $\frac{dx}{x}$ and again integrating, it may readily be found: this second operation gives

$$\frac{(\log x)^3}{2 \cdot 3 \cdot \theta} + \frac{\log x}{1} \underset{1}{c} + \underset{2}{c} = \frac{x-\overset{-1}{x}}{1^2 \sin \theta} - \frac{x^2-\overset{-2}{x}}{2^2 \sin 2\theta} + \frac{x^3-\overset{-3}{x}}{3^2 \sin 3\theta} - \&c.$$

If $x = 1$, $\underset{2}{c} = 0$, put $x = \cos \theta + \sqrt{-1} \sin \theta$, then we have

$$\frac{(\theta\sqrt{-1})^2}{2 \cdot 3 \cdot \theta} + \frac{\theta\sqrt{-1}}{1} \underset{1}{c} = 2\sqrt{-1} \left\{ \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \&c. \right\} = 2\sqrt{-1}\, S\frac{\pm 1}{i^2}^*$$

From this equation the value of c may be found; it is

$$\underset{1}{c} = \frac{1}{\theta} \left\{ \frac{\theta^2}{6} + 2 S \frac{\pm 1}{i^2} \right\}$$

The value of $\underset{1}{c}$ thus found, we have the series

$$\frac{(\log x)^2}{2 \cdot \theta} + \frac{1}{\theta} \left\{ \frac{\theta^2}{6} + 2S\frac{\pm 1}{i^2} \right\} = \frac{x+\overset{-1}{x}}{1 \sin \theta} - \frac{x^2+\overset{-2}{x}}{2 \sin 2\theta} + \frac{x^3+\overset{-3}{x}}{3 \sin 3\theta} - \&c. \quad (5)$$

and

$$\frac{(\log x)^3}{1.2.3.\theta} + \frac{\log x}{\theta} \left\{ \frac{\theta^2}{6} + 2S\frac{\pm 1}{i^2} \right\} = \frac{x-\overset{-1}{x}}{1^2 \sin \theta} - \frac{x^2-\overset{-2}{x}}{2^2 \sin 2\theta} + \frac{x^3-\overset{-3}{x}}{3^2 \sin 3\theta} - \&c. \quad (6)$$

In the first of these put $x = \cos \theta + \sqrt{-1} \sin \theta$, and it becomes

$$-\frac{\theta}{6} + \frac{1}{\theta} S\frac{\pm 1}{i^2} = \frac{\cot \theta}{1} - \frac{\cot 2\theta}{2} + \frac{\cot 3\theta}{3} - \&c. \quad (7)$$

* Throughout the course of this Paper I shall have continual occasion to employ the series $\frac{1}{1^{2n}} - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \&c.$; they can always be expressed by means of the numbers of BERNOUILLI, and the powers of π, and for the sake of brevity I shall always denote them by $S\frac{\pm 1}{i^{2n}}$.

By continuing to multiply (3) by $\frac{dx}{x}$ and integrating, it is easy to perceive that we should arrive at the two following theorems—

$$\frac{(\log x)^{2k}}{1.2..2k.0} + \frac{(\log x)^{2k-2}}{1.2..2k-2}\,c + \&c. + c_{2k-1} =$$

$$\frac{x+x^{-1}}{1^{2k-1}\sin\theta} - \frac{x^2+x^{-2}}{2^{2k-1}\sin 2\theta} + \frac{x^3+x^{-3}}{3^{2k-1}\sin 3\theta} - \&c. \qquad (8)$$

$$\frac{(\log x)^{2k+1}}{1.2..2k+1.0} + \frac{(\log x)^{2k-1}}{1.2..2k-1}\,c + \&c. + \frac{\log x}{1}\,c_{2k-1} =$$

$$\frac{x-x^{-1}}{1^{2k}\sin\theta} - \frac{x^2-x^{-2}}{2^{2k}\sin 2\theta} + \frac{x^3-x^{-3}}{3^{2k}\sin 3\theta} - \&c.$$

the constants c_1, c_3, c_5, &c. may easily be determined from each other, the value of c_1 has been already found, that of c_3 and c_5 are as follows:

$$c_3 = \frac{2}{\theta}\,S\frac{\pm 1}{i^2} + \frac{2\theta}{6}\,S\frac{\mp 1}{i^2} + \frac{7\theta^3}{360}$$

$$c_5 = \frac{2}{\theta}\,S\frac{\pm 1}{i^4} + \frac{2\theta}{6}\,S\frac{\pm 1}{i^4} + 2\,\frac{7\theta^3}{360}\,S\frac{\pm 1}{i^2} + \theta^3\,\frac{31}{5040}$$

a variety of series are deducible from those of (8); I shall only mention two of them;

$$\frac{\cot\theta}{1^{2k-1}} - \frac{\cot 2\theta}{2^{2k-1}} + \frac{\cot 3\theta}{3^{2k-1}} - \&c.$$

and $$\frac{1}{1^{2n}}\cdot\frac{\sin\theta'}{\sin\theta} - \frac{1}{2^{2n}}\cdot\frac{\sin 2\theta'}{\sin 2\theta} + \frac{1}{3^{2n}}\cdot\frac{\sin 3\theta'}{\sin 3\theta} - \&c.$$

Returning to the formulæ (1) and (2) by addition and subtraction, we shall have when n is even

$$(2\sqrt{-1})^n \Sigma^n \left\{ \psi v^{2x+n} + (-1)^n \psi v^{-2x-n} \right\} =$$

$$A_1 \frac{x+x^{-1}}{(\sin\theta)^n} + A_2 \frac{x^2+x^{-2}}{(\sin 2\theta)^n} + A_3 \frac{x^3+x^{-3}}{(\sin 3\theta)^n} + \&c.$$

and when n is an odd number,

$$(2\sqrt{-1})^n \Sigma^n \left\{ \psi v^{2x+n} + (-1)^n \psi v^{-2x-n} \right\} =$$

$$= A \frac{x - \overset{-1}{x}}{(\sin\theta)^n} + A \frac{x^3 - \overset{-3}{x}}{(\sin 2\theta)^n} + A \frac{x^5 - \overset{-5}{x}}{(\sin 3\theta)^n} + \&c.$$

From these expressions it appears, that the reason why we have succeeded in the integrations is, because we had so assumed ψ, that the sum, $\psi v^{2x+n} + \psi v^{-2x-n}$ is a constant quantity; the same success must follow whenever this condition is fulfilled: and hence, we have a method of discovering the sums of a great variety of series, containing the powers of the sines of arcs in arithmetical progression in their denominators, by solving the functional equation $\psi v^{2x+n} + \psi v^{-2x-n} = c$. This is fortunately one of a class whose general solution I have arrived at;* it is

$$\psi v^{2x+n} = \frac{c\varphi v^{2x+n}}{\varphi v^{2x+n} + \varphi v^{-2x-n}} \text{ or } \psi x = \frac{c\varphi x}{\varphi x + \varphi \frac{1}{x}}$$

In the example I have employed n was supposed equal to unity; if this is not the case, we should have found

$$(2\sqrt{-1})^n \Sigma^n (1) = \frac{x \pm \overset{-1}{x}}{(\sin\theta)^n} - \frac{x^3 \pm \overset{-3}{x}}{(\sin 2\theta)^n} + \frac{x^5 \pm \overset{-5}{x}}{(\sin 3\theta)^n} - \&c.$$

If in the functional equation we put $c = 1$, and $\phi x = \tan^{-1} x$, then we have $\psi x = \frac{x}{\pi} \tan^{-1} x$, and

$$(2\sqrt{-1})^n \Sigma^n (1) = \frac{1}{\pi} \left\{ \frac{x \pm \overset{-1}{x}}{1\,(\sin\theta)^n} - \frac{x^3 \pm \overset{-3}{x}}{3\,(\sin 3\theta)^n} + \frac{x^5 \pm \overset{-5}{x}}{5\,(\sin 5\theta)^n} - \&c. \right\}$$

the upper or under sign being used as n, is even or odd; if $n = 1$, the constant is zero; and we have

$$\frac{\pi \log x}{2\theta} = \frac{x - \overset{-1}{x}}{1 \sin\theta} - \frac{x^3 - \overset{-3}{x}}{3 \sin 3\theta} + \frac{x^5 - \overset{-5}{x}}{5 \sin 5\theta} - \&c. \qquad (9)$$

* See Philosophical Transactions for 1817, p. 202.

this may be multiplied by $\frac{dx}{x}$, and integrated any number of times in the same manner as (3), and the results would be

$$\frac{\pi}{2\theta}\cdot\frac{(\log x)^{2k}}{1.2\ldots 2k}+\frac{(\log x)^{2k-2}}{1.2..2k-2}C_1+\&c.+C_{2k-1}=$$
$$=\frac{x+\bar{x}}{1^{2k}\sin\theta}-\frac{x^3+\bar{x}^3}{3^{2k}\sin 3\theta}+\frac{x^5+\bar{x}^5}{5^{2k}\sin 5\theta}-\&c. \qquad (1,1)$$
$$\frac{\pi}{2\theta}\cdot\frac{(\log x)^{2k+1}}{1.2..2k+1}+\frac{(\log x)^{2k-1}}{1.2..2k-1}C_1+\&c.+\frac{\log x}{1}C_{2k-1}=$$
$$=\frac{x-\bar{x}}{1^{2k+1}\sin\theta}-\frac{x^3-\bar{x}^3}{3^{2k+1}\sin 3\theta}+\frac{x^5-\bar{x}^5}{5^{2k+1}\sin 5\theta}-\&c.$$

and these constants may be determined one from the other in the same manner as the former. I shall only give the value of the first, in order to compare the value of the series to which it is equal, with the sum of the same series deduced in another manner.

$$C=\frac{\pi^{\theta}}{1.2}+\frac{2}{\theta}S\frac{\pm 1}{2i+1^{\theta}}=2\left\{\frac{1}{1^{\theta}\sin\theta}-\frac{1}{3^{\theta}\sin 3\theta}+\frac{1}{5^{\theta}\sin 5\theta}-\&c.\right\}(1,2)$$

In order to ascertain the sums of series which contain cosines in their denominators, we must use an artifice which I shall now explain.

Assuming as before $\psi x=A_1x+A_2x^3+A_3x^5+\&c$, and putting v^{2n} for x, we have

$$\psi v^{2n}=A_1v^{2n}+A_2v^{6n}+A_3v^{10n}+\&c.$$

If we were now to integrate this, we should introduce into the denominator of any term $v^{2i}-1$; but we want to introduce the same expression with the signs of both terms positive. If we multiply both sides by $(-1)^{\bar{n}}$ and then integrate, we shall have first

$$(-1)^{\bar{n}}\psi v^{2n}=A_1v^{2n}(-1)^{\bar{n}}+A_2v^{6n}(-1)^{\bar{n}}+A_3v^{10n}(-1)^{\bar{n}}+\&c.$$

And since $\Delta\,(-1)^{\frac{z}{2}}\,v^{2iz} = -\,(-1)^{\frac{z}{2}}\,v^{2iz+2i} - (-1)^{\frac{z}{2}}\,v^{2iz} = -\,(v^{2i}+1)\,(-1)^{\frac{z}{2}}\,v^{2iz}$ we find

$$\Sigma(-1)^{z}\,v^{2iz} = -\,\frac{v^{2iz}}{v^{2i}+1}$$

Integrating each term separately, we have

$$\Sigma(-1)^{\frac{z}{2}}\,\psi v^{iz} = -\left\{ A_1 \frac{v^{iz}(-1)^{\frac{z}{2}}}{v^{2i}+1} + A_2 \frac{v^{iz}(-1)^{\frac{z}{2}}}{v^{2i}+1} + A_3 \frac{v^{iz}(-1)^{\frac{z}{2}}}{v^{2i}+1} + \&c. \right\}$$

Let this integration be repeated n times, it will give

$$(-1)^{n}\,\Sigma^{n}(-1)^{\frac{z}{2}}\,\psi v^{iz} = A_1 \frac{v^{iz}(-1)^{\frac{z}{2}}}{(v^{2i}+1)^{n}} + A_2 \frac{v^{iz}(-1)^{\frac{z}{2}}}{(v^{2i}+1)^{n}} + A_3 \frac{v^{iz}(-1)^{\frac{z}{2}}}{(v^{2i}+1)^{n}} + \&c.$$

Let $v = \cos\theta\sqrt{-1}\,\sin\theta$, and $z + \frac{\pi}{2}$ for z; this becomes

$$(-2)^{n}\,\Sigma^{n}(-1)^{\frac{z}{2}-\frac{n}{2}}\,\psi v^{2iz+n} = (-1)^{\frac{z}{2}-\frac{n}{2}} \left\{ A_1 \frac{v^{iz}}{(\cos\theta)^{n}} + A_2 \frac{v^{iz}}{(\cos 2\theta)^{n}} + A_3 \frac{v^{iz}}{(\cos 3\theta)^{n}} + \&c. \right\}$$

And finally,

$$(-2)^{n}\,(-1)^{n}\,\Sigma^{n}(-1)^{\frac{z}{2}}\,\psi v^{2iz+n} = A_1 \frac{v^{iz}}{(\cos\theta)^{n}} + A_2 \frac{v^{iz}}{(\cos 2\theta)^{n}} + A_3 \frac{v^{iz}}{(\cos 3\theta)^{n}} + \&c.$$

$$= A_1 \frac{v^{z}}{(\cos\theta)^{n}} + A_2 \frac{v^{2z}}{(\cos 2\theta)^{n}} + A_3 \frac{v^{3z}}{(\cos 3\theta)^{n}} + \&c. \qquad (1,8)$$

The integrations here indicated will, as in a former instance, generally surpass the powers of analysis in its present state; but a contrivance similar to that which has been already stated, will in many cases elude the difficulty: the artifice consists in investigating another similar series arranged according to the descending powers of the variable, integrating it in the same manner as we have that marked (1,8), and adding these two results, we shall in many cases have a function which is integrable, and the two series become equal in the case of $z = 1$. By commencing with the descending

series $\psi v^{2n+x} = A_1 v^{n+x} + A_2 v^{n+x} + A_3 v^{n+x} + \&c.$ multiplying by $(-1)^n$ and integrating, we shall get the expression

$$(-2)^n(-1)^n \Sigma^n(-1)^n \psi v^{2n-n} = A_1 \frac{v^{n+x}}{(\cos\theta)^n} + A_2 \frac{v^{n+x}}{(\cos 2\theta)^n} + A_3 \frac{v^{n+x}}{(\cos 3\theta)^n} - \&c.$$

$$= A_1 \frac{x^n}{(\cos\theta)^n} + A_2 \frac{x^n}{(\cos 2\theta)^n} + A_3 \frac{x^n}{(\cos 3\theta)^n} + \&c. \qquad (1,4)$$

There occur very few cases in which it is possible to execute the integrations in (1,3) and (1,4); by adding the two together, we have

$$(-2)^n(-1)^n \Sigma^n(-1)^n \left\{\psi v^{2n+x} + \psi v^{2n-x}\right\} = A_1 \frac{x^n+x^n}{(\cos\theta)^n} + A_2 \frac{x^n+x^n}{(\cos 2\theta)^n}$$

$$+ A_3 \frac{x^n+x^n}{(\cos 3\theta)^n} + \&c. \qquad (1,5)$$

Here we may observe that the new series is exactly double either of the others (1,3) or (1,4) when $x=1$; also, that the integration on the left side can be executed any number of times, whenever $\psi v^{2n+x}\,\psi v^{-2n-x}$ is a constant quantity; the forms of the function ψ, which fulfil this condition, have already been given. Let $\psi x = \frac{x}{1+x}$, then $A_1 = 1$, $A_2 = -1$, $A_3 = 1$, &c. and since $\psi v^{2n+x} + \psi v^{-2n-x} = 1$, we have

$$(-2)^n(-1)^n \Sigma^n(-1)^n = \frac{x+x^2}{(\cos\theta)^n} - \frac{x^n+x^n}{(\cos 2\theta)^n} + \frac{x^n+x^n}{(\cos 3\theta)^n} + \&c.$$

These integrations are easily executed; and commencing with $n=1$, we have

$$-2\,.\,(-1)^n \frac{(-1)^n}{-1-1} + 2b(-1)^n = 1 + 2b(-1)^n = \frac{x+x^2}{\cos\theta} - \frac{x^n+x^n}{\cos 2\theta} + \&c$$

In order to determine the constant b, put $x = \cos\theta + \sqrt{-1}\sin\theta$; then, since x in that case becomes $\frac{1}{2}$, we have

$$1 + 2b\sqrt{-1} = 2 - 2 + 2 - 2 + \&c. = 1$$

hence $b = 0$, and we have

$$1 = \frac{x+\overline{x}^{-1}}{\cos\theta} - \frac{x^2+\overline{x}^{-2}}{\cos 2\theta} + \frac{x^3+\overline{x}^{-3}}{\cos 3\theta} - \&c. \qquad (1,6)$$

Continuing to integrate, it will be found that all the constants are zero, and we shall arrive at the following theorem;

$$1 = \frac{x+\overline{x}^{-1}}{(\cos\theta)^n} - \frac{x+\overline{x}^{-2}}{(\cos 2\theta)^n} + \frac{x^3+\overline{x}^{-3}}{(\cos 3\theta)^n} - \&c. \qquad (1,7)$$

Let $x = -x$, then it becomes

$$-1 = \frac{x+\overline{x}^{-1}}{(\cos\theta)^n} + \frac{x^2+\overline{x}^{-2}}{(\cos 2\theta)^n} + \frac{x^3+\overline{x}^{-3}}{(\cos 3\theta)^n} + \&c. \qquad (1,8)$$

Putting $x = 1$ in both these, we have

$$\frac{1}{2} = \frac{1}{(\cos\theta)^n} - \frac{1}{(\cos 2\theta)^n} + \frac{1}{(\cos 3\theta)^n} \&c. \qquad (1,9)$$

$$-\frac{1}{2} = \frac{1}{(\cos\theta)^n} + \frac{1}{(\cos 2\theta)^n} + \frac{1}{(\cos 3\theta)^n} + \&c. \qquad (2,1)$$

I propose in the next place to determine the value of the series

$$\frac{1}{1^{2k}(\cos\theta)^n} - \frac{1}{2^{2k}(\cos 2\theta)^n} + \frac{1}{3^{2k}(\cos 3\theta)^n} - \&c.$$

This may be accomplished by multiplying (1,7) by $\frac{dx}{x}$, and integrating; this operation, being performed on it 2^k times, will produce the series whose sum is required; the first integration gives

$$\frac{\log x}{1} + C = \frac{x-\overline{x}^{-1}}{1(\cos\theta)^n} - \frac{x^2-\overline{x}^{-2}}{2(\cos 2\theta)^n} + \frac{x^3-\overline{x}^{-3}}{3(\cos 3\theta)^n} - \&c.$$

If $x = 1$, $C = 0$, the second operation gives

$$\frac{(\log x)^2}{1\cdot 2} + \frac{C}{2} = \frac{x+\overline{x}^{-1}}{1^2(\cos\theta)^n} - \frac{x^2+\overline{x}^{-2}}{2^2(\cos 2\theta)^n} - \&c.$$

If $x = 1$ $C_{2,n} = \frac{2}{1^2(\cos\theta)^n} - \frac{2}{2^2(\cos 2\theta)^n} + \&c.$

In order to determine $C_{2,n}$ put $x = \cos\theta + \sqrt{-1}\sin\theta$, then we have

$$-\frac{\theta^2}{2}+c_{2,n}=\frac{2}{1^{2}(\cos\theta)^{n-1}}-\frac{2}{2^{2}(\cos\theta)^{n-1}}+\&c.=c_{2,n-1}.$$

The equation $c_{2,n}-c_{2,n-1}=\frac{\theta^2}{2}$ being integrated, gives

$$c_{2,n}=\frac{n}{2}\cdot\frac{\theta^2}{2}+b$$

If $n=0$ $\quad c_{2,0}=b=\frac{2}{1^2}-\frac{2}{2^2}+\frac{2}{3^2}-\&c.=2S\frac{\pm 1}{i^2}$

Hence $\quad c_{2,n}=\frac{n}{1}\cdot\frac{\theta^2}{2}+2S\frac{\pm 1}{i^2}$ and

$$\frac{(\log x)^2}{1.2}+\frac{n}{1}\cdot\frac{\theta^2}{2}+2S\frac{\pm 1}{i^2}=\frac{x+x^{-1}}{1^2(\cos\theta)^n}-\frac{x^2+x^{-2}}{2^2(\cos 2\theta)^n}+\frac{x^3+x^{-3}}{3^2(\cos 3\theta)^n}\ \&c.\quad (2,2)$$

These integrations being repeated, we shall arrive at the two following expressions:

$$\frac{(\log x)^{2k}}{1.2\ldots 2k}+\frac{(\log x)^{2k-2}}{1.2\ldots 2k-2}c_{2,n}+\&c.+\frac{(\log x)^2}{1.2}c_{2k-2,n}+c_{2k,n}=$$

$$=\frac{x+x^{-1}}{1^{2k}(\cos\theta)^n}-\frac{x^2+x^{-2}}{2^{2k}(\cos 2\theta)^n}+\frac{x^3+x^{-3}}{3^{2k}(\cos 3\theta)^n}+\&c.\qquad (2,3)$$

$$\frac{(\log x)^{2k+1}}{1.2\ldots 2k+1}+\frac{(\log x)^{2k-1}}{1.2\ldots 2k-1}c_{2,n}+\&c.\ c_{2k,n}=$$

$$\frac{x-x^{-1}}{1^{2k+1}(\cos\theta)^n}-\frac{x^2-x^{-2}}{2^{2k+1}(\cos 2\theta)^n}+\frac{x^3-x^{-3}}{3^{2k+1}(\cos\theta)^n}-\&c.$$

It now becomes necessary to determine the value of $c_{2k,n}$, which is equal to twice the sum of the series we are investigating; for if $x=1$

$$c_{2k,n}=\frac{2}{1^{2k}(\cos\theta)^n}-\frac{2}{2^{2k}(\cos 2\theta)^n}+\frac{2}{3^{2k}(\cos 3\theta)^n}-\&c.$$

For this purpose put in the first of the equations (2,3) $x=\cos\theta+\sqrt{-1}\sin\theta$, then the series on the right hand is equal to $c_{2k,n-1}$, and we have for determining $c_{2k,n}$ the equation of finite differences.

$$\frac{(\theta\sqrt{-1})^{2k}}{1.2\ldots 2k}+\frac{(\theta\sqrt{-1})^{2k-2}}{1.2\ldots 2k-2}c_{2,n}+\frac{(\theta\sqrt{-1})^{2k-4}}{1.2\ldots 2k-4}c_{4,n}+\&c.+c_{2k,n}=c_{2k,n-1}$$

In order to integrate this equation, let us suppose $c_{2k,n}$ to represent the co-efficient of r^{2k} in the developement of $\frac{f(r)}{(ar)^n}$

where $a(r) = 1 + Ar^2 + Br^4 + Cr^6 + \&c.$

then $\frac{f(r)}{(ar)^n} = 1 + rc_{2,n} + r^2c_{4,n} + r^3c_{6,n} + \&c.$

If this be multiplied by ar, it becomes

$$\frac{f(r)}{(ar)^{n-1}} = 1 + r^2c_{2,n} + r^4c_{4,n} + r^6c_{6,n} + r^2A + r^4Ac_{2,n} + r^6Ac_{4,n} + r^4B + r^6Bc_{2,n} + r^6C$$

But the co-efficient of r^{2k} in this series is equal to $c_{2k,n-1}$ hence

$$c_{2k,n} + Ac_{2k-2,n} + Bc_{2k-4,n} + \&c. = c_{2k,n-1}$$

This equation will become the one in question, if we make

$$A = \frac{(b\sqrt{-1})^2}{1.2} \quad B = \frac{(b\sqrt{-1})^4}{1.2.3.4} \quad C = \frac{(b\sqrt{-1})^6}{1.2\ldots6} \quad \&c.$$

This produces

$$c_{2k,n} + \frac{(b\sqrt{-1})^2}{1.2}c_{2k-2,n} + \frac{(b\sqrt{-1})^4}{1.2.3.4}c_{2k-4,n} + \&c. = c_{2k,n-1}$$

We have now only to determine the form of $f(r)$, and this may be easily accomplished since the values of $c_{2k,0}$ are known; for if we put $n = 0$

$$f(r) = 1 + r^2c_{2,0} + r^4c_{4,0} + r^6c_{6,0} + \&c.$$
$$= 1 + 2r^2S\tfrac{\pm 1}{i^2} + 2r^4S\tfrac{\pm 1}{i^4} + 2r^6S\tfrac{\pm 1}{i^6} + \&c.$$

Therefore $c_{2k,n}$ is equal to the co-efficient of r^{2k} in the developement of

$$\frac{1 + 2r^2S\frac{\pm 1}{i^2} + 2r^4S\frac{\pm 1}{i^4} + \&c.}{(\cos r\theta)^n}$$

Or if $(\cos r\theta)^{-n} = 1 + A'_n\theta^2r^2 + B'_n\theta^4r^4 + C'_n\theta^6r^6 + \&c.$ then

$$c_{2k,n} = 2S\frac{\pm 1}{i^{2k}} + 2A'_n \theta^2 S\frac{\pm 1}{i^{2k-2}} + 2B'_n \theta^4 S\frac{\pm 1}{i^{2k-4}} + 2C'_n \theta^6 S\frac{\pm 1}{i^{2k-6}} + \&c.$$

The quantity $c_{2k,n}$ may now be considered as completely determined, since it only depends on the co-efficients of $(\cos\theta)^{-n}$, and the series marked by $S\frac{\pm 1}{i^{2k}}$, both which quantities are known; the latter being given by the powers of π and numbers of BERNOUILLI, whilst the values of the former in functions of n are given by LEGENDRE, in his Exercise de Calcul Integral, vol. iii. art. 149, 153.

In (2,3) let $x = 1$, and we have

$$\frac{1}{2}c_{2k,n} = \frac{1}{1^{2k}(\cos\theta)^n} - \frac{1}{2^{2k}(\cos 2\theta)^n} + \frac{1}{3^{2k}(\cos 3\theta)^n} - \&c. \qquad (2,4)$$

And if we put $x = v$ in the other series, it becomes

$$\frac{\theta}{2}\left\{\frac{(-\theta^2)^k}{1.2\ldots 2k+1} + \frac{(-\theta^2)^{k-1}}{1.2\ldots 2k-1}c_{2,n} + \&c. + \frac{(-\theta^2)}{1.2.3}c_{2k-2,n} + \frac{1}{1}c_{2k,n}\right\} =$$

$$= \frac{\sin\theta}{1^{2k+1}(\cos\theta)^n} - \frac{\sin 2\theta}{2^{2k+1}(\cos 2\theta)^n} + \frac{\sin 3\theta}{3^{2k+1}(\cos 3\theta)^n} - \&c. \qquad (2,5)$$

If $n = 1$ this series becomes

$$\frac{\tan\theta}{1^{2k+1}} - \frac{\tan 2\theta}{2^{2k+1}} + \frac{\tan 3\theta}{3^{2k+1}} - \&c. \qquad (2,6)$$

The series (2,4) may be changed into another, which contains sines both in the numerator and the denominator, for it is equal to

$$\frac{1}{2}c_{2k,n} = \frac{1}{1^{2k}}\left(\frac{\sin\theta}{\sin\theta.\cos\theta}\right)^n - \frac{1}{2^{2k}}\left(\frac{\sin 2\theta}{\sin 2\theta.\cos 2\theta}\right)^n + \frac{1}{3^{2k}}\left(\frac{\sin 3\theta}{\sin 3\theta.\cos 3\theta}\right)^n - \&c.$$

But this becomes, since $\sin\theta.\cos\theta = \frac{1}{2}\sin 2\theta$

$$\frac{2}{2^{n+1}}c_{2k,n} = \frac{2}{1^{2k}}\left(\frac{\sin\theta}{\sin 2\theta}\right)^n - \frac{1}{2^{2k}}\left(\frac{\sin 2\theta}{\sin 4\theta}\right)^n + \frac{1}{3^{2k}}\left(\frac{\sin 3\theta}{\sin 6\theta}\right)^n - \&c. \qquad (2,7)$$

By applying the theorems (*a*), (*b*), (*c*), and (*d*) to the series whose sums we have now investigated, we shall arrive at the value of many others which contain the powers of tangents

and co-tangents in arithmetical progression, thus (2,9) combined with (*a*) will produce

$$\frac{1}{2}\left\{1-\frac{k}{1}+\frac{k.k-1}{1.2}-\&c.\right\}=\frac{(\sin\theta)^{2l}}{(\cos\theta)^{2l}}-\frac{(\sin 2\theta)^{2l}}{(\cos 2\theta)^{2l}}+\frac{(\sin 3\theta)^{2l}}{(\cos 3\theta)^{2l}}-\&c.$$

But the left side of this equation is equal to $\frac{1}{2}(1-1)^k=0$. Hence

$$0=\frac{(\sin\theta)^{2l}}{(\cos\theta)^{2l}}-\frac{(\sin 2\theta)^{2l}}{(\cos 2\theta)^{2l}}+\frac{(\sin 3\theta)^{2l}}{(\cos 3\theta)^{2l}}-\&c. \qquad (2,8)$$

And if $n=2k$, this produces a series of tangents

$$0=(\tan\theta)^{2l}-(\tan 2\theta)^{2l}+(\tan 3\theta)^{2l}-\&c. \qquad (2,9)$$

By means of the theorems already referred to, we may introduce into the numerators of each term of the series (2,4) the even powers of the sines of the same arcs whose co-sines occur in the denominator: putting $l=\frac{m}{2}$, we shall have

$$\frac{1}{2}\left\{C_{2k,n}-\frac{l}{1}C_{2k,n-2}+\frac{l.l-1}{1.2}C_{2k,n-4}-\&c.\right\}=$$
$$=\frac{(\sin\theta)^{2l}}{1^{2l}(\cos\theta)^{2l}}-\frac{(\sin 2\theta)^{2l}}{2^{2l}(\cos 2\theta)^{2l}}+\frac{(\sin 3\theta)^{2l}}{3^{2l}(\cos 3\theta)^{2l}}-\&c. \qquad (3,1)$$

And if $n=2l$, this becomes

$$\frac{1}{2}\left\{C_{2k,2l}-\frac{l}{1}C_{2k,2l-2}+\frac{l.l-1}{1.2}C_{2k,2l-4}-\&c.\right\}=$$
$$=\frac{(\tan\theta)^{2l}}{1^{2l}}-\frac{(\tan 2\theta)^{2l}}{2^{2l}}+\frac{(\tan 3\theta)^{2l}}{3^{2l}}-\&c. \qquad (3,2)$$

If we call the sum of the series (2,5) $A_{k,n}$, and if we apply to it the formula (*b*), we shall have

$$A_{k,n}-\frac{l}{1}A_{k,n-2}+\frac{l.l-1}{1.2}A_{k,n-4}-\&c.=$$
$$=\frac{(\sin\theta)^{2l+1}}{1^{2k+1}(\cos\theta)^{n}}-\frac{(\sin 2\theta)^{2l+1}}{2^{2k+1}(\cos 2\theta)^{n}}+\frac{(\sin 3\theta)^{2l+1}}{3^{2k+1}(\cos 3\theta)^{n}}-\&c. \qquad (3,3)$$

And if $n=2l+1$, this becomes

$$A_{k,2l+1}-\frac{l}{1}A_{k,2l-1}+\&c.=$$
$$=\frac{(\tan\theta)^{2l+1}}{1^{2k+1}}-\frac{(\tan 2\theta)^{2l+1}}{2^{2k+1}}+\frac{(\tan 3\theta)^{2l+1}}{3^{2k+1}}-\&c. \qquad (3,4)$$

In the equation (1,4) if we make $\psi x = \tan^{-1} x$, we shall find

$$(-2)^n(-1)^n\Sigma^n(-1)^{-n}\left\{\tan^{-1} v^{2x+n} + \tan^{-1} v^{-2x-n}\right\} =$$

$$= (-2)^n(-1)^n\Sigma^n(-1)^{-n}\cdot\frac{\pi}{4} = \frac{x+x^{-1}}{1(\cos\theta)^n} - \frac{x^3+x^{-3}}{3(\cos 3\theta)^n} + \frac{x^5+x^{-5}}{5(\cos 5\theta)^n} - \&c.$$

If the integrations here indicated are performed, it will be found that all the constants vanish, and ultimately that

$$\frac{\pi}{2} = \frac{x+x^{-1}}{1(\cos\theta)^n} - \frac{x^3+x^{-3}}{3(\cos 3\theta)^n} + \frac{x^5+x^{-5}}{5(\cos 5\theta)^n} - \&c. \qquad (3,5)$$

If $x = 1$

$$\frac{\pi}{4} = \frac{1}{1(\cos\theta)^n} - \frac{1}{3(\cos 3\theta)^n} + \frac{1}{5(\cos 5\theta)^n} - \&c. \qquad (3,6)$$

If we multiply (3,5) by $\frac{dx}{x}$ and integrate, we have

$$\frac{\pi}{2}\log x + C = \frac{x-x^{-1}}{1^2(\cos\theta)^n} - \frac{x^3-x^{-3}}{3^2(\cos 3\theta)^n} + \frac{x^5-x^{-5}}{5^2(\cos 5\theta)^n} - \&c. \qquad (3,7)$$

If $x = 1$ $C = o$, let $x = \cos\theta' + \sqrt{-1}\sin\theta'$, then we have

$$\frac{n\theta'}{2} = \frac{\sin\theta'}{1^2(\cos\theta)^n} - \frac{\sin 3\theta'}{3(\cos 3\theta)^n} + \frac{\sin 5\theta'}{5(\cos 5\theta)^n} - \&c. \qquad (3,8)$$

The equation (3,7) may be multiplied any number of times by $\frac{dx}{x}$, and integrated; and the constants thus introduced may be determined in the same manner as those of the equations (2,3); these operations will give the values of series of the following form:

$$\frac{x+x^{-1}}{1^{2k+1}(\cos\theta)^n} - \frac{x^3+x^{-3}}{3^{2k+1}(\cos 3\theta)^n} + \frac{x^5+x^{-5}}{5^{2k+1}(\cos 5\theta)^n} - \&c.$$

$$\frac{x-x^{-1}}{1^{2k}(\cos\theta)^n} - \frac{x^3-x^{-3}}{3^{2k}(\cos 3\theta)} + \frac{x^5-x^{-5}}{5^{2k}\cos 5\theta} - \&c.$$

Numerous other series might be found by satisfying the equation $\psi x + \psi\frac{1}{x} = 1$, whose sums would be given by this process; and if instead of putting $\frac{1}{x}$ for x, we had substituted ωx for x, where the function ω is determined by the equation $\omega^2 x = x$, many others would be discovered. This artifice is

only a particular case of a much more general principle, which is of use in discovering certain values of the variable, in which a series admits of summation, but which generally is not expressible in finite terms. The principle is as follows: let K denote any operation, such as integration, either with respect to differential or finite differences, or any other operation, provided only $K(X+Y)=KX+KY$. Now let

$$K\psi x = A_1 x + A_2 x^2 + A_3 x^3 +$$

Put $\alpha x, \alpha^2 x, \ldots \alpha^{n-1} x$ for x, and the results will be

$$K\psi \alpha x = A_1 \alpha x + A_2 \alpha x^2 + A_3 \alpha x^3 +$$

$$K\psi \alpha^2 x = A_1 \alpha^2 x + A_2 \alpha^2 x^2 + A_3 \alpha^2 x^3 +$$

&c. &c.

$$K\psi \alpha^{n-1} x = A_1 \alpha^{n-1} x + A_2 \alpha^{n-1} x^2 + A_3 \alpha^{n-1} x^3 +$$

By adding all these together, we shall have $K\{\psi x + \psi \alpha x + \psi \alpha^2 x +$ &c. $\psi \alpha^{n-1} x\}$ equal to a series whose general term is $A_n\{x + \alpha x + \alpha^2 x +$ &c. $+ \alpha^{n-1} x\}$. Now supposing we cannot perform the operation denoted by K on the function ψx, yet if ψ is of such a form that $\psi x + \psi \alpha x +$ &c. $+ \psi \alpha^{n-1} x$ is equal to a function on which the operation K can be executed, then calling this new function ψ_1, we shall have

$$K\psi_1 x = SA_i\{x^i + \alpha x^i + \ldots \alpha^{n-1} x^i\}$$

And if α is such a function that $\alpha^n x = x$, a great variety of forms for ψ may be found, which will satisfy that condition. Now let x be determined by the equation $x = \alpha x$, and r being any root of this, we have $r = \alpha r = \alpha^2 r = \&c. = \alpha^{n-1} r$.

Consequently our equation becomes

$$K\psi_1 x = nSA_i r^i = n\{A_1 r + A_2 r^2 + A_3 r^3 + A_4 r^4 +$$ &c.$\}$

provided we put r for x after the operation K is executed; that is, we have found the values of the series

$$A_1x + A_2x^2 + A_3x^3 + \&c.$$

in the particular cases of x which satisfy the equation $ax = x$.

PART II.

I shall now explain another method of deducing the sums of a variety of series, which comprehend amongst them all those which are contained in the former part of this paper; it rests fundamentally on the following formulæ, which have long been known:

$$0 = 1^{2n} - 2^{2n} + 3^{2n} - 4^{2n} + \&c.$$
$$\tfrac{1}{2} = \cos\theta - \cos 2\theta + \cos 3\theta - \&c.$$
$$\tfrac{1}{2} = \cos\theta + \cos 2\theta + \cos 3\theta + \&c.$$
$$0 = 1^{2n+1} - 3^{2n+1} + 5^{2n+1} - \&c.$$

It is unnecessary to give proofs of these and other similar ones which have been frequently noticed, as they may be very easily demonstrated.

Let fx be any function of x developeable in even powers of x, then $f(x) = A + Bx^2 + Cx^4 + Dx^6 + \&c.$

Divide both sides by x^{2k}, then it becomes

$$\frac{f(x)}{x^{2k}} = \frac{A}{x^{2k}} + \frac{B}{x^{2k-2}} + \&c. + K + Lx^2 + Mx^4 + \&c.$$

For x, put successively $1x$, $2x$, $3x$, $4x$, &c. and let the alternate series be taken negatively; these being arranged under each other, we have

$$+\frac{f(x)}{1^{2k}x^{2k}} = +\frac{A}{1^{2k}x^{2k}} + \frac{B}{1^{2k-2}x^{2k-2}} + \&c. + K + Lx^2 1^2 + Mx^4 1^4 +$$
$$-\frac{f(2x)}{2^{2k}x^{2k}} = -\frac{A}{2^{2k}x^{2k}} - \frac{B}{2^{2k-2}x^{2k-2}} - \&c. - K - Lx^2 2^2 - Mx^4 2^4 -$$

$$+\frac{f(3x)}{3^{2k}x^{2k}} = +\frac{A}{3^{2k}x^{2k}} + \frac{B}{3^{2k-2}x^{2k-2}} + \&c. + K + Lx^2 3^2 + Mx^4 3^4 +$$
$$-\frac{f(4x)}{4^{2k}x^{2k}} = -\frac{A}{4^{2k}x^{2k}} - \frac{B}{4^{2k-2}x^{2k-2}} - \&c. - K - Lx^2 4^2 - Mx^4 4^4 -$$
$$\&c. \qquad \&c. \qquad \&c.$$

If we add the vertical columns, we shall have on the left side of the equation the series

$$\frac{1}{x^{2k}}\left\{\frac{f(x)}{1^{2k}} - \frac{f(2x)}{2^{2k}} + \frac{f(3x)}{3^{2k}} - \frac{f(4x)}{4^{2k}} + \&c.\right\}$$

and the right side of the equation consists of three kinds of terms, those which contain negative powers of x, one which does not contain x, and the remaining ones which contain positive powers of x. With respect to these last, they are all of the form $Qx^{2i}\left\{1^{2i} - 2^{2i} + 3^{2i} - 4^{2i} + \&c.\right\}$; and as the series which multiplies Qx^{2i} is equal to zero, all those vertical columns which contain even powers of x will vanish: the term which is independent on x is

$$K - K + K - K + \&c. = \tfrac{1}{2}K$$

and those terms which contain negative powers of x, may be represented by the expression $S\frac{\pm 1}{i^{2k}}$. All the vertical columns being summed, we shall have the equation

$$\frac{f(x)}{1^{2k}} - \frac{f(2x)}{2^{2k}} + \frac{f(3x)}{3^{2k}} - \&c. = AS\frac{\pm 1}{i^{2k}} + BS\frac{\pm 1}{i^{2k-2}} +$$
$$+ CS\frac{\pm 1}{i^{2k-4}} + \&c. + \tfrac{1}{2}K \qquad (A)$$

As the operations by which we have arrived at this expression have been given at length, it will be unnecessary to repeat them with the slight modifications which would be required for cases nearly similar. Thus, if we suppose the function fx developeable according to the odd powers of x, and if we divide both sides of the equation by x^{2k+1} and

repeat the same process we have already explained, we shall arrive at the following theorem:

$$\frac{f(x)}{1^{2k+1}} - \frac{f(2x)}{2^{2k+1}} + \frac{f(3x)}{3^{2k+1}} - \frac{f(4x)}{4^{2k+1}} + \&c. =$$

$$= Ax^{}S\frac{\pm 1}{i^{2k}} + Bx^{3}S\frac{\pm 1}{i^{2k-2}} + \&c. + \tfrac{1}{2}K \qquad (B)$$

Let $f(\theta)$ be any function of θ developeable in the form

$$f(\theta) = A + B\cos\theta + C\cos 2\theta + D\cos 3\theta \ \&c.$$

a very similar process to that which has been already explained will give the

$$f(\theta) - f(2\theta) + f(3\theta) - \&c. = \frac{f(o)}{2} \qquad (C)$$

and if $f(\theta) = A\cos\theta + B\cos 2\theta + C\cos 3\theta + \&c.$ a similar course will produce the equation

$$f(\theta) + f(2\theta) + f(3\theta) + \&c. = -\tfrac{1}{2}f(o) \qquad (D)$$

If a function is developeable in even powers of x, then its second function is developeable in the same manner, and so are all its higher functions; therefore if f_1 and f are two functions developeable in even powers of x, such that

$$fx = A' + B'x^{2} + C'x^{4} + \&c.$$

$$f_1 f^{n}x = A + Bx^{2} + Cx^{4} + \&c.$$

Then (A) will become

$$\frac{f_1 f^{n}(x)}{1^{2n}} - \frac{f_1 f^{n}(2x)}{2^{2n}} + \frac{f_1 f^{n}(3x)}{3^{2n}} - \&cc. =$$

$$= AS\frac{\pm 1}{i^{2k}} + BS\frac{\pm 1}{i^{2k-2}} + \&c. + \tfrac{1}{2}K \qquad (E)$$

These theorems marked (A), (B), (C), and (D), although they possess a very great degree of generality, are not entirely without restriction; it appears at first sight that they are applicable to *all* functions which have the prescribed condition of being expansible in even powers of the variable; such was my opinion of them when I first discovered them; but

several results which were evidently incorrect, soon convinced me that some limitation existed, of whose nature I was not aware: it was not until some years after, that I found out the cause of the fallacies which had perplexed me; and still more recently, I discovered that the series on whose sum their truth or falsehood depended, might be expressed by a definite integral. By applying the criterion, which I shall presently explain, we cut off a great variety of series whose sums are erroneously given by the method in question; whether this criterion does not exclude some series whose sums are correctly given, is a point which I do not consider yet completely decided; the difficulties to which the application of acknowledged principles have in this instance conducted us, appear worthy of the attention of mathematicians. A more strict method might have been pursued in determining the sum of that part of the series which is neglected; but this in general leads to such differential equations, as cannot afford us much assistance. I have, however, given one example of this method, and I have shown that when the part which had been neglected, as being apparently equal to zero (but which is in fact a finite quantity,) is added to the sum furnished by the *method of expanding horizontally and summing vertically*, the true value of the series results. This confirms the explanation I have given of the reason of the apparent failure of that method.

It will be sufficient to point out the cause which leads to error, and to determine the conditions on which its existence depends for one only of the series; suitable modifications of the reasoning will readily suggest themselves for the others. I shall therefore, at present, consider the theorem (A). If

we turn to the process employed in its investigation, we may remark, that the vertical column $Lx^{\circ}(1^{\circ} - 2^{\circ} + 3^{\circ} - 4^{\circ} + \&c.)$ has been neglected, because the series which enters into it as a factor is equal to zero; so also the vertical column $Mx^{\circ}(1^{\circ} - 2^{\circ} + 3^{\circ} - \&c.)$ is neglected for the same reason, and similarly for all the remaining vertical columns. Now, although it would be perfectly correct to omit any one, or even any finite number of these vertical columns, as being multiplied by a factor equal to zero, yet it is not legitimate to neglect an infinite number of terms, each multiplied by zero, unless it can be proved that the sum of all the terms so multiplied is not an infinite quantity: this, then, is the latent cause of the false results at which I arrived at the commencement of these enquiries. I shall now explain how they may be obviated, or rather how to assign the condition on which the truth of the theorems just deduced depend. We have considered the series of terms

$$Lx^{\circ}(1^{\circ} - 2^{\circ} + 3^{\circ} - \&c.) + Mx^{\circ}(1^{\circ} - 2^{\circ} + 3^{\circ} - \&c.) + \\ + Mx^{\circ}(1^{\circ} - 2^{\circ} + 3^{\circ} - \&c.) + \&c.$$

as equal to zero. Any one of the series which here multiply the powers of x, may be considered as arising from the series

$$1^{n}y - 2^{n}y^{2} + 3^{n}y^{3} + \&c.$$

When $y = 1$, call this series $K_{n}(y)$, and instead of making $y = 1$, let $y = 1 + o$, which differs from unity by the infinitely small quantity o; then we shall have

$$K_{n}(1 + o) = c_{1,n} + c_{2,n}o + c_{3,n}o^{2} + \&c.$$

Where

$$c_{1,n} = \sigma = 1^{2n} - 2^{2n} + 3^{2n} - \&c. \quad c_{2,n} = 1^{2n+1} - 2^{2n+1} + 3^{2n+1} - \&c.$$

by substituting this value of $K_{n}(1 + o)$ in the series we had neglected, we shall find

$$\begin{aligned}
&\left\{ Lx^{a}c_{1,1} + Mx^{c}c_{1,2} + Nx^{e}c_{1,3} + \&c. \right\} \\
&+ o \left\{ Lx^{a}c_{2,1} + Mx^{c}c_{2,2} + Nx^{e}c_{2,3} + \&c. \right\} \\
&+ o^{3} \left\{ Lx^{a}c_{3,1} + Mx^{c}c_{3,2} + Nx^{e}c_{3,3} + \&c. \right\} \\
&+ o^{5} \left\{ Lx^{a}c_{4,1} + Mx^{c}c_{4,2} + Nx^{e}c_{4,3} + \&c. \right\} \\
&+ \quad \&c. \qquad\qquad \&c.
\end{aligned}$$

The first line vanishes on account of the value of $c_{1,n}$, and since o is an infinitesimal, the second line will be larger than the sum of all the rest, provided the multiplicators of the powers of o are finite; if therefore the series $Lx^{a}c_{2,1} + Mx^{c}c_{2,2} + Nx^{e}c_{2,3} + \&c.$ is finite since it is multiplied by o, we may neglect the whole of the above expression: our next step must be to determine whether the series

$$\begin{aligned}
&Lx^{a}\left\{ 1^{3} - 2^{3} + 3^{3} - \&c. \right\} + Mx^{c}\left\{ 1^{3} - 2^{3} + 3^{3} - \&c. \right\} + \\
&\qquad + Nx^{e}\left\{ 1^{7} - 2^{7} + 3^{7} - \&c. \right\} + \&c.
\end{aligned}$$

is finite or infinite. It has been observed by EULER, that the following relations exist between the direct and reciprocal powers of the natural numbers.

$$\begin{aligned}
1 - 2 + 3 - 4 + \&c. &= + 2\,\frac{1}{\pi^{2}}\left\{ 1 + \frac{1}{3^{2}} + \frac{1}{5^{2}} + \&c. \right\} \\
1^{3} - 2^{3} + 3^{3} - 4^{3} + \&c. &= - 2\,\frac{1.2.3}{\pi^{4}}\left\{ 1 + \frac{1}{3^{4}} + \frac{1}{5^{4}} + \&c. \right\} \\
1^{5} - 2^{5} + 3^{5} - 4^{5} + \&c. &= + 2\,\frac{1\ldots5}{\pi^{6}}\left\{ 1 + \frac{1}{3^{6}} + \frac{1}{5^{6}} + \&c. \right\} \\
1^{7} - 2^{7} + 3^{7} - 4^{7} + \&c. &= - 2\,\frac{1\ldots7}{\pi^{8}}\left\{ 1 + \frac{1}{3^{8}} + \frac{1}{5^{8}} + \&c. \right\} \\
\&c. \qquad &\qquad \&c.
\end{aligned}$$

These latter being substituted for their equals, we have

$$\begin{aligned}
&- 2Lx^{a}\frac{1.2.3}{\pi^{4}}\left(\frac{1}{1^{4}} + \frac{1}{3^{4}} + \frac{1}{5^{4}} + \&c.\right) + 2Mx^{c}\frac{1.2.3.4.5}{\pi^{6}}\left(\frac{1}{1^{6}} + \frac{1}{3^{6}} + \right. \\
&\left. + \frac{1}{5^{6}}\ \&c.\right) - 2Nx^{e}\frac{1\ldots7}{\pi^{8}}\left(\frac{1}{1^{8}} + \frac{1}{3^{8}} + \frac{1}{5^{8}} + \&c.\right) + \&c.
\end{aligned}$$

The series which now multiplies each term is in all cases

less than 2, consequently this expression will be finite or infinite, according as the series

$$-Lx^{\alpha}\frac{1.2.3}{x^{\alpha}}+Mx^{\alpha}\frac{1...5}{x^{\alpha}}-Nx^{\alpha}\frac{1...7}{x^{\alpha}}+\&c.$$

is so or the contrary: the product $1.2...n$ may be expressed by means of a definite integral, thus:

$$1.2...n=\int dv\left(\log\frac{1}{v}\right)^{n}\qquad\begin{bmatrix}v=0\\v=1\end{bmatrix}$$

These products being replaced by the integral, we have

$$\frac{1}{x^{\alpha}}\int dv\left\{-L\left(\frac{x}{v}\right)^{2}\left(\log\frac{1}{v}\right)^{3}+M\left(\frac{x}{v}\right)^{4}\left(\log\frac{1}{v}\right)^{5}-N\left(\frac{x}{v}\right)^{6}\left(\log\frac{1}{v}\right)^{7}+\&c.\right\}\quad\begin{bmatrix}v=0\\v=1\end{bmatrix}$$

$$=\frac{1}{x^{3}}\int dv\log\frac{1}{v}\cdot\left\{-L\left(\frac{x}{v}\log\frac{1}{v}\right)^{2}+M\left(\frac{x}{v}\log\frac{1}{v}\right)^{4}-N\left(\frac{x}{v}\log\frac{1}{v}\right)^{6}+\&c.\right\}$$

Now in order to determine the sum of this series, which evidently depends on the function $f(x)$, let us assume

$$\frac{f(x)}{x^{2k}}-\frac{A}{x^{2k}}-\frac{B}{x^{2k-2}}-\&c.-K=\chi(x)$$

then we have

$$\chi(x)=Lx^{4}+Mx^{6}+Nx^{8}+\&c.$$

And if we put $\frac{x\sqrt{-1}}{v}\log\frac{1}{v}$ instead of x, it becomes

$$\chi\left(\frac{x\sqrt{-1}}{v}\log\frac{1}{v}\right)=-L\left(\frac{x}{v}\log\frac{1}{v}\right)^{2}+M\left(\frac{x}{v}\log\frac{1}{v}\right)^{4}-N\left(\frac{x}{v}\log\frac{1}{v}\right)^{6}+\&c.$$

And the sum of the series in question is

$$\frac{1}{x^{3}}\int dv\log\frac{1}{v}\cdot\chi\left(\frac{x\sqrt{-1}}{v}\log\frac{1}{v}\right)\qquad\begin{bmatrix}v=0\\v=1\end{bmatrix}$$

If therefore this definite integral is finite, the theorem (A) will give correct results. A more convenient form for integration may however be obtained by the following consideration; the series

$$L\frac{x^{4}}{x^{4}}.1.2.3+M\frac{x^{6}}{x^{4}}1...5+N\frac{x^{8}}{x^{4}}1...7+\&c.$$

will always be finite, if the following series

$$A+B\left(\frac{x}{v}\right)^{3}1.2.3+C\left(\frac{x}{v}\right)^{4}1.2..5+D\left(\frac{x}{v}\right)^{6}1.2..7+\&c.$$

is finite, because this series when prolonged to the terms

$Mx^n + Nx^{n+1} +$ &c. will have its terms each greater than the corresponding terms of the other series. Now this series is equal to

$$\int dv\, f\left(\frac{\pi\sqrt{-1}}{\varpi}\log\frac{1}{v}\right) \qquad \begin{bmatrix} v=o \\ v=1 \end{bmatrix} \qquad (F)$$

I shall now apply some of the theorems to the investigation of the sums of series, and then explain a method which (when the equations to which it leads can be solved) will in all cases render them correct. And first let $f(\theta) = (\cos\theta)^n = (1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} -$ &c.$)^n$ this series is capable of being expanded into another, which also proceeds according to the even powers of θ; first let $k = o$, then comparing this with (A), we have

$$\tfrac{1}{2} = (\cos\theta)^n - (\cos 2\theta)^n + (\cos 3\theta)^n - \text{&c.} \qquad (3.9)$$

Let us now examine if the definite integral is finite, it is in this case.

$$\int dv\left\{\cos\frac{\theta\sqrt{-1}}{\varpi}\log\frac{1}{v}\right\}^n = \int dv\left\{\frac{-\frac{\theta}{\varpi}\log\frac{1}{v} + \frac{\theta}{\varpi}\log\frac{1}{v}}{2}\right\}^n =$$

$$= \int dv\left[\frac{\frac{\theta}{\varpi}\log\frac{v+v}{2} - \frac{\theta}{\varpi}}{2}\right]^n$$

$$= \frac{1}{2^n}\left\{\frac{1}{1-n\frac{\theta}{\varpi}} + \frac{n}{1}\cdot\frac{1}{1+(2-n)\frac{\theta}{\varpi}} + \frac{n.n-1}{1.2}\cdot\frac{1}{1+(4-n)\frac{\theta}{\varpi}} + \text{&c.}\right\} \qquad \begin{bmatrix} v=o \\ v=1 \end{bmatrix}$$

If n is a whole positive number we already know that the series (3.9) is correct; if n is a fraction the series which expresses the value of the definite integral is finite for all values of θ, except such as are contained in $\theta = -\frac{\varpi}{2i-n}$ i being any whole number; if n is negative, then the series is finite for all positive values of θ; it appears then that whatever be the value of n if θ is positive, we have

$$\frac{1}{2} = \frac{1}{(\cos\theta)^n} - \frac{1}{(\cos 2\theta)^n} + \frac{1}{(\cos 3\theta)^n} - \&c.$$

From the theorem (A) we may readily determine the value of the series

$$\frac{1}{1^{2l}(\cos\theta)^n} - \frac{1}{2^{2l}(\cos 2\theta)^n} + \frac{1}{3^{2l}(\cos 3\theta)^n} - \&c.$$

For let $f(\theta) = \frac{1}{(\cos\theta)^n} = A' + \underset{n}{A'}\theta^n + \underset{n}{B'}\theta^n + \underset{n}{C'}\theta^n + \&c.$

And the sum of the series required will be

$$S\frac{\pm 1}{i^{2k}} + \underset{n}{A'}\theta^n S\frac{\pm 1}{i^{2k-2}} + \underset{n}{B'}\theta^n S\frac{\pm 1}{i^{2k-4}} + \&c. + \theta^{2l}\underset{n}{K'}\cdot\frac{\pi}{2}$$

Which is precisely the same sum as we have already found in (2,4), except that we now find that it applies to fractional or surd values of n, as well as to whole numbers.

Let us next suppose $f(\theta) = (\tan\theta)^{2l+1} = T_1\theta^{2l+1} + T_3\theta^{2l+3} + T_5\theta^{2l+5} + \&c.$ this give the series

$$\frac{(\tan\theta)^{2l+1}}{1^{2k+1}} - \frac{(\tan 2\theta)^{2l+1}}{2^{2k+1}} + \frac{(\tan 3\theta)^{2l+1}}{3^{2k+1}} - \&c. =$$

$$= T_1\theta S\frac{\pm 1}{i^{2(k-l)}} + T_3\theta^3 S\frac{\pm 1}{i^{2(k-l-1)}} + \&c.$$

The definite integral in this case being

$$\int dv \left\{\tan\frac{\theta\sqrt{-1}}{\pi}\log\frac{1}{v}\right\}^{2l+1} = \frac{1}{(\sqrt{-1})^{2l+1}}\int dv\left\{\frac{-1+v^{2\frac{\theta}{\pi}}}{1+v^{2\frac{\theta}{\pi}}}\right\}^{2l+1} \quad \left[\begin{matrix} v=0 \\ v=1 \end{matrix}\right]$$

$$= \frac{1}{(\sqrt{-1})^{2l+1}}\left\{A + \frac{B}{1+2\frac{\theta}{\pi}} + \frac{C}{1+4\frac{\theta}{\pi}} + \frac{D}{1+6\frac{\theta}{\pi}} + \&c.\right\}$$

Since $\frac{-1+v^{2\frac{\theta}{\pi}}}{1+v^{2\frac{\theta}{\pi}}} = A + Bv^{2\frac{\theta}{\pi}} + Cv^{4\frac{\theta}{\pi}} + \&c.$ this is always finite if θ is positive, because it is less than $A + B + C + \&c.$ which is equal to zero.

In the former part we could only determine the value of the series

$$\frac{1}{(\sin\theta)^n} - \frac{1}{(\sin 2\theta)^n} + \frac{1}{(\sin 3\theta)^n} - \&c.$$

when n is an even number, nor will the method now employed enable us to find its value when n is odd; the reason of this is that

$$(\sin\theta)^{-n} = \left(\frac{\theta}{1} - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \&c.\right)^{-n}$$

is developable in a series proceeding according to the even powers of θ only when n is an even number, if n is odd, it proceeds according to the odd powers.

The theorems contained in this second part are applicable to a very extensive class of series which have not, I believe, yet been considered. In (A) let $f(\theta) = \cos^2\theta = \cos(\cos\theta)$ and putting $k = o$ we have

$$\frac{\cos 1}{2} = \cos^2\theta - \cos^2 2\theta + \cos^2 3\theta - \&c. \qquad (4,1)$$

The definite integral which is the criterion of the truth of this value, is

$$\int dv \cos\frac{v\frac{\theta}{\varpi} + v\frac{-\theta}{\varpi}}{2} = A + \frac{B}{1+2\frac{\theta}{\varpi}} + \frac{C}{1+4\frac{\theta}{\varpi}} + \frac{D}{1+6\frac{\theta}{\varpi}} + \&c. \quad \left[\begin{matrix} v=o \\ v=1 \end{matrix}\right]$$

$$+ \frac{B}{1-2\frac{\theta}{\varpi}} + \frac{C}{1-4\frac{\theta}{\varpi}} + \frac{D}{1-6\frac{\theta}{\varpi}} + \&c.$$

And this is always finite unless θ is an even submultiple of ϖ; if we make $k = 1$ and $k = \varpi$, we shall have the following theorems, which are true with the same restrictions.

$$\cos 1.\ S\frac{\pm 1}{i^2} + \frac{\sin 1}{2}\cdot\frac{\theta^2}{2} = \frac{\cos^2\theta}{1^2} - \frac{\cos^2 2\theta}{2^2} + \frac{\cos^2 3\theta}{3^2} - \&c. \qquad (4,2)$$

$$\cos 1.\ S\frac{\pm 1}{i^4} + \theta^2\frac{\sin 1}{2}S\frac{\pm 1}{i^2} - \frac{\theta^4}{2}\left(\frac{\sin 1}{1.2.3.4} + \frac{\cos 1}{8}\right) =$$

$$= \frac{\cos^2\theta}{1^4} - \frac{\cos^2 2\theta}{2^4} + \frac{\cos^2 3\theta}{3^4} - \&c. \qquad (4,3$$

If $f(\theta) = (\cos^2\theta)^n$, since this is capable of being developed

in a series proceeding according to the even powers of θ, we shall have if $k = o$

$$\frac{(\cos^{n-1}\theta)^m}{2} = (\cos^n\theta)^m - (\cos^n 2\theta)^m + (\cos^n 3\theta)^m - \&c. \qquad (4.4)$$

and the definite integral is finite in this case, whenever θ and π are incommensurable: we may therefore in the same circumstance have the value of the series

$$\frac{(\cos^n\theta)^m}{1^{2i}} - \frac{(\cos^n 2\theta)^m}{2^{2i}} + \frac{(\cos^n 3\theta)^m}{3^{2i}} - \&c.$$

The theorems marked (A) and (B) in this paper correspond with that marked (12) in Mr. HERSCHEL'S memoir "On various points of Analysis," printed in the Philosophical Transactions for 1814; with the first of these it coincides when n is an even number, and with the second when it is an odd one: the theorem alluded to is

$$S\left\{\frac{(-1)^{x+1}}{x^{2i}} \cdot \frac{f(\sqrt{-1}x\theta) + (-1)^n f(\sqrt{-1}-x\theta)}{2}\right\} = {}^{i}L(2) \cdot a_n\theta^n + {}^{i}L(2) \cdot a_{n-2}\theta^{n-2} + \&c.$$

where ${}^{i}L = 1 - 1 + 1 - \&c. = \frac{1}{2}\,{}^{i}L(2) = \frac{(2^{2i-1}-1)\pi^{2i}}{1.2\ldots 2i} B_{2i-1}$

Now this latter expression is the value of those series which I have expressed by $S\frac{\pm 1}{i^{2i}}$. Both methods give the same result, and as that result is very frequently erroneous, I shall confirm the truth of the explanation I have offered, by shewing in a particular case, that if the sum of that part of the series which had been neglected as being equal to zero, is found and added to the other part, the result will no longer be erroneous: the example I shall examine is the series

$$\frac{1}{1+t^2} - \frac{1}{1+2^2t^2} + \frac{1}{1+3^2t^2} - \frac{1}{1+4^2t^2} + \&c.$$

In Mr. HERSCHEL'S theorem, making $f(s) = a_0 + a_1 t + a_2 t^2 + \&c. = \frac{1+bt\sqrt{-1}}{1+b^2t^2}$ we have $a_0 = 1$, and the equation becomes

$$S\left\{\frac{(-1)^{x+1}\left\{\frac{1+x\theta\sqrt{-1}}{1+x^2\theta^2}+\frac{1-x\theta\sqrt{-1}}{1+x^2\theta^2}\right\}}{2}\right\}=\text{'L}(2).1$$

or $\quad S(-1)^{x+1}\frac{1}{1+x\theta^2}=\frac{1}{2}$ or

$$\frac{1}{2}=\frac{1}{1+\theta^2}-\frac{1}{1+2^2\theta^2}+\frac{1}{1+3^2\theta^2}-\text{&c.} \qquad (4.4)$$

The same series being summed by the theorem (A) in this paper gives the same result; but then the theorem alluded to only declares this to be the true value in case a certain series is finite, which series is

$$+\theta^2.1.2.3+\theta^4.1.2.3.4.5+\theta^6.1.2..7+\text{&c.}$$

but this series can only be finite when θ is actually equal to zero: the method which I have explained in this paper points out that the equation

$$\frac{1}{2}=\frac{1}{1+\theta^2}-\frac{1}{1+2^2\theta^2}+\frac{1}{1+3^2\theta^2}-\text{&c.}$$

can only be depended on when $\theta=o$, in that case it is known to be correct. I have already stated that the reason why the value of the series so found is incorrect, is that the series $-\theta^2(1^2-2^2+\text{&c.})+\theta^4(1^4-2^4+\text{&c.})-\theta^6(1^6-2^6+\text{&c.})+\text{&c.}$ has been neglected because the coefficient of each term is zero. I shall now proceed to investigate the sum of this series, and shall prove that it is equal to a finite function of θ: let

$$y=c^2(1^2\varepsilon^x-2^2\varepsilon^{2x}+\text{&c.})+c^4(1^4\varepsilon^x-2^4\varepsilon^{2x}+\text{&c.})+\text{&c.}$$

then y is equal to the sum of the series whose value we are seeking; if $c=\theta\sqrt{-1}$ and $x=o$, differentiate y twice relative to x, and multiply by c^2, and we find

$$c^2\frac{d^2y}{dx^2}=c^2(1^2\varepsilon^x-2^2\varepsilon^{2x}+\text{&c.})+c^4(1^4\varepsilon^x-2^4\varepsilon^{2x}+\text{&c.})+\text{&c.}$$

Hence the equation for determining y is

$$c^2\frac{d^2y}{dx^2}-y=-c^2(1^2\varepsilon^x-2^2\varepsilon^{2x}+\text{&c.})=-c^2\frac{\varepsilon^x-\varepsilon^{2x}}{(1+\varepsilon^x)^3}$$

And the value of y is

$$y = \frac{c}{2}\left\{ \varepsilon^{\frac{\pi}{c}} \int \varepsilon^{-\frac{\pi}{c}} \frac{\varepsilon^{2x}-\varepsilon^{x}}{(1+\varepsilon^{x})^{2}}\, dx - \varepsilon^{-\frac{\pi}{c}} \int \varepsilon^{\frac{\pi}{c}} \frac{\varepsilon^{2x}-\varepsilon^{x}}{(1+\varepsilon^{x})^{2}} \right\}$$

These integrals must be taken between the limits $x = -\infty$ and $x = o$, putting $\varepsilon^{x} = v$ this equation is changed into

$$y = \frac{c}{2}\left\{ v^{\frac{1}{c}} \int \frac{v-1}{(1+v)^{2}}\, v^{-\frac{1}{c}}\, dv - v^{-\frac{1}{c}} \int \frac{v-1}{(1+v)^{2}}\, v^{\frac{1}{c}}\, dv \right\}$$

where the limits of v are $v = o$ $v = 1$, in the latter integral put $v = \frac{1}{u}$, and we have

$$y = \frac{c}{2}\left\{ v^{\frac{1}{c}} \int \frac{v-1}{(1+v)^{2}}\, v^{-\frac{1}{c}} dv + u^{\frac{1}{c}} \int \frac{u-1}{(1+u)^{2}}\, u^{-\frac{1}{c}} du \right\} \quad \left\{ \begin{matrix} v = o & u = 1 \\ v = 1 & u = \infty \end{matrix} \right\}$$

but this is equal to $\frac{c}{2} v^{\frac{1}{c}} \int \frac{v-1}{(1+v)^{2}}\, v^{-\frac{1}{c}}\, dv$ between the limits $v = o$ and $v = \infty$, which is equal to $-\frac{\pi}{2c \sin \frac{\pi}{c}}$ hence

$$y = -\frac{\pi}{2c \sin \frac{\pi}{c}} = c^{1}(1^{1} - 2^{1} + \&c.) + c^{2}(1^{2} - 2^{2} + \&c.) + c^{3}(1^{3} - 2^{3} + \&c.) + \&c.$$

If $c = \theta\sqrt{-1}$ we have

$$-\frac{\pi}{\theta\left\{ \varepsilon^{\frac{\pi}{\theta}} - \varepsilon^{-\frac{\pi}{\theta}} \right\}} = -\theta^{1}(1^{1} - 2^{1} + \&c.) + \theta^{2}(1^{2} - 2^{2} + \&c.) - \theta^{3}(1^{3} - 2^{3} + \&c.) + \&c.$$

This being added to $\frac{1}{2}$ the value given by the theorem (A) produces

$$\frac{1}{2} - \frac{\pi}{\theta\left\{ \varepsilon^{\frac{\pi}{\theta}} - \varepsilon^{-\frac{\pi}{\theta}} \right\}} = \frac{1}{1+\theta^{1}} - \frac{1}{1+2^{1}\theta^{1}} + \frac{1}{1+3^{1}\theta^{1}} - \frac{1}{1+4^{1}\theta^{1}} + \&c.$$

which is the same value that EULER had assigned to this series.

From the value which has been found for y, or for the series

$$c^{1}(1^{1} - 2^{1} + \&c.) + c^{2}(1^{2} - 2^{2} + \&c.) + \&c.$$

I am inclined to conclude that although the series $1^{1.2} - 2^{1.2} +$

$3^{2n} -$ &c. is equal to zero for any finite value of n, yet that when n is infinite, the sum of this series is also infinite.

It was my intention to have produced from several of the series whose sums have been found in this Paper, the values of several continued products; but the length to which it has already extended will prevent me from more than merely noticing, that many very curious ones will present themselves by integrating the series whose sums have been given.

Since this Paper was written, in a conversation with M. Poisson, I mentioned one of the series which it contained, and remarked, that the principle employed led to many erroneous results; that gentleman observed, that many years before he had been led to series nearly similar, in endeavouring to integrate the equations representing the planetary motions, by means of series arranged according to some other functions of the time than the usual ones of the sines and cosines: he obligingly showed me some of his papers relating to this subject, in one of which was a series which in a particular case became the one I had mentioned; the mode of investigation by which he had arrived at these series he had however laid aside, because it rested on the sums of the diverging series $1^{2n} - 2^{2n} + 3^{2n} -$ &c. on which he observed we cannot depend. To the same distinguished analyst I am indebted for some farther information on this subject. M. Poisson was one of the commissioners appointed by the Institute of France to examine the manuscripts which were left by Lagrange, amongst these was one entitled "a method of summing series," which depended on the values of the same diverging

series as those used by M. POISSON and myself; unfortunately it is very short, and its illustrious author does not appear to have resumed the subject: possibly the erroneous values which it gives for the sum of certain series might have caused him to reject it.

C. BABBAGE.

March 25, 1819.

XIX. *On the optical and physical properties of Tabasheer.* By David Brewster, *LL.D. F.R.S. Lond. and Edin. In a Letter to the Right Hon. Sir* Joseph Banks, *Bart. G.C.B. P.R.S. &c. &c. &c.*

Read May 6, 1819.

My Dear Sir,

The substance called Tabasheer, has been long used as a medicine in Turkey, Syria, Arabia, and Hindostan. It was first made generally known in Europe by Dr. Patrick Russell, who published in the Philosophical Transactions, for 1790, a very interesting account of its natural history, and of the process by which it seems to be formed. From his enquiries it appears, that this substance is found in the cavities of the bamboo, the *Arundo bambos* of Linnæus; and that it exists originally in the state of a transparent fluid, which acquires by degrees the consistency of a mucilage resembling honey, and is afterwards converted by gradual induration into a white solid, called Tabasheer. From the analysis of Mr. Macie (now Mr. Smithson), it appeared to be "perfectly identical with common siliceous earth."*

The celebrated traveller M. Humboldt, discovered the same substance in the bamboos which grow to the west of Pinchincha, in South America, and a portion of what he brought to Europe in 1804, was analyzed by Fourcroy and Vauquelin, who found it to consist of 70 parts of silex, and 30 of potash and lime.†

* See *Philosophical Transactions* for 1791, p. 368.

† *Mémoires de l'Institut*, Tom. vi. p. 382.

About two years ago my friend, Dr. KENNEDY, received from India, a considerable quantity of tabasheer, a portion of which he presented to the Royal Society of Edinburgh. From this portion I took two or three fragments, with the view of ascertaining if it possessed any crystalline structure, but having found that it had no particular action upon polarised light, I was not led to any farther examination of its physical properties. In the course, however, of a series of experiments upon the phosphorescence of minerals, I was surprised to observe, that the tabasheer emitted a light when placed upon a hot iron, more intense than most of the phosphorescent minerals. This unexpected property induced me to resume the investigation, and having received, through the kindness of Dr. KENNEDY, an additional supply of tabasheer, I was enabled to examine it with peculiar care in all its optical and physical relations.

Among the pieces of tabasheer which I have examined, there appear to be three different kinds. The first has a milky transparency, transmitting a *yellowish*, and reflecting a *bluish white* light. It is easily broken between the fingers, and has a sort of aerial and unsubstantial texture, entirely different from any other solid substance. The second kind is more opaque, and harder than the first, having a slight degree of translucency at the edges; and the third kind is perfectly opaque, resembling a piece of stucco or chalk, or the opaque subsulphate of alumine.

If we form two parallel faces upon a piece of transparent tabasheer, by grinding it on a plate of smooth but unpolished glass, we shall be able to see objects through it with perfect distinctness, although no polish is induced upon the surfaces.

By slightly wetting the tabasheer, it loses all its transparency, and assumes the appearance of a piece of chalk; but if we immerse it in water a great quantity of gas is disengaged, the edges become more transparent than before, and a small white ball appears in the centre, which gradually diminishes till the transparency has extended itself throughout the whole mass. The same effect takes place with the second kind of tabasheer; but the third kind, though it disengages gas like the rest, never loses its chalky opacity.

The property of becoming more transparent by the expulsion of air and the absorption of water, is one which the tabasheer possesses in common with hydrophanous opal; but the faculty of retaining a considerable degree of transparency when it is dry, and its pores filled with air; and the still more extraordinary faculty of becoming quite opaque by the absorption of a small quantity of water, are possessed by no other known substance in nature, and indicate a singularity of structure which it becomes interesting to investigate.

When the pores of hydrophane are filled with air, the difference between the refractive power of the air and the solid substance is so great, that the light is scattered in every direction by refraction, and the mass is consequently white and opaque. As the tabasheer disengages a much greater quantity of air than the hydrophane, its pores must be more numerous, and therefore the transmission of the light, so as to form a perfect image, indicates either an extreme feebleness of refractive power, or some singularity in the form and construction of the pores themselves.

In order to determine this, I formed a prism of tabasheer, and upon measuring its refractive power, I found it to be

extremely low, and to vary in different specimens, as stated in the following table.

	Index of Refraction.
Transparent tabasheer from Vellore*	1.1115
Transparent tabasheer from Nagpore	1.1454
Another specimen of the same -	1.1503
A third specimen of the same -	1.1535
A harder and more opaque specimen,	1.1825
Water - - -	1.3358
Flint glass - - -	1.600
Sulphur - - -	2.115
Phosphorus - - -	2.224
Diamond - - -	2.470

Hence it follows, that tabasheer has a lower refractive power than all other bodies, whether solid or fluid; and that it holds an intermediate place between *water* and the *gases*. This extraordinary result which, as it were, insulates tabasheer from all known substances, will enable us to give an explanation of some of its most remarkable properties.

The singular nature of this substance will appear in a still more striking point of view, by comparing its absolute refracting power with that of other bodies. The low refractive power of air, when contrasted with that of water, and of water when contrasted with that of solid bodies, obviously arises from the great difference of their specific gravities, and not from any peculiar action upon light. If we call R the

* This specimen, with which I was favoured by Dr. HOPE, formed part of the tabasheer sent by Dr. PATRICK RUSSELL to the Royal Society, in 1790. It was yellowish by reflected light, and so extremely tender, that I was obliged to polish it upon the softest silk.

absolute refractive power of any body, m its index of refraction, and S its specific gravity, we shall have $R = \frac{m^2-1}{S}$.

By means of this formula I have computed the following table, for the purpose of showing the peculiar nature of tabasheer, and the general progression in the absolute refractive powers of other bodies.

TABLE, *showing the absolute refractive powers of tabasheer and other bodies.*

Tabasheer	-	976.1*
Sulphate of barytes		3829.43
Air, - -	BIOT	4530.

* A distinguished member of the Royal Society, whose opinion is entitled to the highest consideration, has kindly stated to me, that in estimating the absolute refractive power of tabasheer, I should have taken its specific gravity at about 0.66 in place of 2.4, which would have given an absolute refractive power not so marvellously different from that of other bodies.

My reason for retaining the original number of 976.1, is, that the result obtained by using a specific gravity of 0.66 would have been a *theoretical, and not an experimental result.* A body which sinks in water must have a density greater than unity; and I am supported by the high authority of Mr. CAVENDISH, and also by that of Mr. SMITHSON, in considering the specific gravity of tabasheer as nearly 2.412. Capillary spaces which are so large as to contain water, and even the thickest oils and varnishes, can never be regarded as forming a part of the body, though, in the present case, it is highly probable that these spaces modify the action of the solid particles in the manner which I have described.

Although the idea that tabasheer is quartz expanded till its specific gravity is reduced to about 0.66, holds out a sort of general explanation of its singular properties; yet when we consider that *hydrophane*, which is also quartz expanded till it is capable of absorbing water, is actually opaque; and when we recollect also that the refractive power of bodies does not always diminish when they expand, as is proved by the experiments of ALBERT EULER on heated glass, and by the circumstance that the point of maximum density in water is not indicated by a maximum of refractive power, its optical properties must continue to appear as marvellous as before.

Quartz, -	MALUS	5414.37
Calcareous spar,	MALUS	6423.5
Flint glass, *lowest*		7238.
Ruby -		7388.8
Brazilian topaz -		7586.7
Water, -	MALUS	7845.7
Flint glass, highest		8735.
Carbonate of potash		10227.
Chromate of lead		10436.
Nitre -		11962.
Muriate of soda -		12086.
Bees wax, -	MALUS	13308.1
Diamond, -	MALUS	13964.5
Sulphur - -		22000.
Phosphorus -		28857.
Hydrogen -		29964.
——— -		31862.

It appears from the preceding table, that tabasheer, which was placed between water and air, in the comparison of their indices of refraction, is now not only the lowest of all substances in its absolute refractive power, but it is so extremely low as to be separated by a great interval from them all. The very high refractive power of sulphur, phosphorus and hydrogen, and the great interval between diamond and sulphur are very remarkable, and indicate that hydrogen may enter largely into the composition of sulphur and phosphorus.

I now saturated the prism of tabasheer with *water* and with oil of cassia in succession, and I found that, in the first case, its refractive power was raised to 1.4012 higher than that of water, and in the second case to 1.6493, a little greater than

that of oil of cassia. The oil communicated to the prism a very yellow tinge, and was retained by it for a very long time.

Tabasheer readily imbibes all the volatile and fat oils, and indeed all the fluids that I have tried. The essential oils are quickly absorbed, and with the exception of oil of cassia are as quickly evaporated, while the fat oils are slowly drawn in and remain a long time in its pores; and in all these cases an opacity is produced by a partial absorption exactly as in the case of water.

When the oils or other fluids have a colour of their own, or are tinged with any colouring matter, the tabasheer exhibits a similar tint, so that it is easy to communicate to it any colour that we please. From a solution of acetate of copper, it acquires the colour of the emerald; from any of the oils coloured with anchusa root, it receives the tints of the ruby; from oil of beech nut, the colour of the chryso-beryl; from sulphuric acid, that of the pink topaz; and from malic acid, that of the Brazilian topaz. These different colours may be all discharged by exposing the tabasheer to a red heat, and thus expelling the absorbed fluid to which they owe their origin.

The opaque tabasheer, which retains its opacity when its pores are filled with water, acquires the most beautiful transparency from the absorption of oil of beech nut; and it is curious to observe a substance like chalk, and consisting apparently of a number of particles in a state of accidental aggregation, converted into a transparent mass, which the light freely penetrates in every direction. Having saturated a large piece of this kind of tabasheer with oil of beech nut, coloured with anchusa root, I laid it on a mass of lead of a lower temperature than that of the room. The oil instantly

retired from the surface of the tabasheer into its interior, and the transparent mass became opaque like a piece of red brick. Upon removing it from the lead into its former temperature, the oil returned to the surface, and the tabasheer resumed its transparency. If, on the other hand, we place it in a higher temperature than that of the room, a part of the oil will be discharged, and when it is brought back to its first temperature, it will become opaque like a piece of brick. Even when a small part of the oil remains, the transparency may be readily restored by the application of a sufficiently high degree of heat. The phenomena which have now been described admit of a satisfactory explanation from the difference between the expansion of the oil and that of the tabasheer; but the effect appears to be too great to arise from this cause, and I am rather disposed to ascribe it to a variation in the capacity of the tabasheer for the oil by a change of temperature.

In order to observe the nature of the penumbral boundary which might be supposed to separate the opaque from the transparent part, if they could be both rendered visible in the same mass, I saturated the largest piece of tabasheer that I had with the coloured oil, and having discharged a good deal of it by heat, it became of course opaque. I now held to the flame of a candle one of its extremities, which immediately became diaphanous, and the transparency gradually pervaded the opaque mass. As soon as the opacity disappeared, I allowed one extremity of it to cool; the transparency immediately disappeared at that part, and the opacity gradually advanced like a black cloud, till the whole was overshadowed by the retreat of the oil into the interior of

the mass. In both these cases the penumbra, which separated the opaque and transparent portions, had a ragged or branching appearance when seen by a microscope, as if the oil had been shooting into crystals during the progress of the opacity, and as if these crystals had been dissolving during its retreat.

Upon examining the appearance of the tabasheer when it had the colour of red brick, after the discharge of a portion of its oil, I was surprised to observe a beautiful veined structure, exactly like that of the agate, the veins being sometimes parallel, sometimes inflected, and sometimes curved. In some specimens the veins were alternately opaque and translucent, and in others red and white, as if one set of strata had a greater capacity for the oil than the rest. This effect is almost universal; but as soon as the oil is completely discharged, the veined structure entirely disappears, and the whole mass assumes the homogeneous appearance of chalk.

In order to observe the circumstances under which the chalky tabasheer became transparent by the absorption of oils, I cut four plates out of the same piece, and immersed them separately in oil of cassia, alcohol, water, and oil of beech nut. The plates that had absorbed the three first of these fluids remained quite opaque, but the plate that was placed in the oil of beech nut gradually acquired a translucency by the rapid extrication of air. After a certain time it appeared to be covered with scratches and small opaque portions; but these appearances, which arose from remaining vesicles of air, vanished by degrees. By the application of a microscope, I observed the air form itself into globules in the interior of the tabasheer, which slowly advanced to the edge of it, and at last escaped into the oil. After a lapse of

nearly two hours the greater part of the air was extricated, and the expulsion of the remainder was quickly effected by a gentle heat; but the transparency was of that imperfect kind, which results from the union of two bodies of different refractive powers. By increasing the heat, the tabasheer became more transparent, and at a certain temperature it could scarcely be seen in the oil in which it was placed. When the heat was still farther augmented, it became more and more opaque, and a corresponding opacity was induced by cooling it down as much below the temperature of maximum transparency.

When pure tabasheer is boiled for any length of time, or is brought to a red or a white heat, it suffers no change either in its colour or in its optical and physical properties: if we wrap it, however, in a piece of paper, and set the paper on fire, the tabasheer becomes either black, or brownish black, and the black colour increases in depth by the repetition of the experiment. When immersed in water, it disengages the included air, but with less rapidity than before; and when it is broken and pounded, its fracture and its powder are black.

If the blackened tabasheer is brought to a red heat, it is restored to its primitive whiteness, and resumes all its former properties; but if the heat is considerably below redness, some specimens acquire a slight transparency, and a dark slaty blue colour, shading in some places into whiteness. When slightly wetted in this state, it becomes chalky white; with a greater portion of water it becomes black; and with a still greater portion, it becomes again transparent. If we break a piece of tabasheer a few days after it has received a deep

black tint, we shall find that it has often a fine ash grey colour, which becomes deeply black when wetted, and afterwards resumes its primitive tint when dry.

As the blackness communicated to the tabasheer is not produced by heat alone, or by any particular method of cooling, it cannot arise from any mechanical change, similar to that which THENARD observed in the cooling of melted phosphorus; and there is reason to believe that it does not arise from the absorption of any sooty matter thrown off during the combustion of the paper, as the blackened tabasheer almost always recovers a certain degree of whiteness, and imbibes water almost as freely as in its original state. With the view of discovering if the blackness was owing to any vegetable matter in the tabasheer, I repeated the operation of blackening it and restoring its colour by heat about 50 times; but after all these operations, it became black as readily as at first. The specimen that had undergone these changes had increased in hardness and lustre, and had the appearance of the finest Indian ink. Upon breaking it in two, the fracture was perfectly black, but assumed a dark blue colour by exposure to the air; and upon putting a drop of pure water upon the blue fracture, it was instantly converted into a deep and glossy black.

When the pure tabasheer was exposed to a white heat for several hours, and was then burned in paper, it received its black colour as before. When it was held in the flame of alcohol or of carburetted hydrogen, it was merely stained in a slight degree, which was probably owing to the intensity of the heat, which may have discharged the black tint as soon as it was formed.

I was now anxious to see the effect produced upon tabasheer by the absorption of iodine gas, and for this purpose I took several opaque and transparent fragments, and having saturated some of them with moisture, and left others dry, I placed them in different glass tubes, which were hermetically sealed after a portion of iodine had been introduced. Previous to the application of heat the tabasheer assumed a yellow tinge, which deepened into a pale orange, and a veined structure appeared in one of the fragments. When the iodine was converted into gas by heat, the colour of the fragments grew more and more red; the transparent pieces were like garnets, and the opaque ones like fragments of red brick; and after standing two or three days the opaque pieces became perfectly transparent. The iodine vapour, therefore, seems to have taken the place of the water in the wetted fragments, and of the common air in the dry ones; and it appears to have retained its gaseous form within the tabasheer, when the external gas had returned to the solid state. Upon taking the tabasheer out of the glass tubes, the iodine was slowly expelled, a yellow tinge appeared even after 30 hours exposure to the air, and it was not entirely removed by immersing the fragments in water.

The difference in the properties of the opaque and the transparent tabasheer, rendered it desirable to have accurate measures of their specific gravities, and I have been fortunate enough to procure them to a great degree of exactness, through the kindness of my friend, Mr. JAMES JARDINE, who obtained for me the following results.

	Grains.
No. I. Seven pieces of opaque tabasheer, weighed in air. - -	6 65
The same pieces when thoroughly soaked in water, weighed in air -	14,10
The same weighed in water at the temperature of 52° - -	3,42

Hence the specific gravity of the parcel when dry, is 2.059
And the specific gravity of the parcel when wet, is 1.320

	Grains
No. II. Several small fragments of transparent tabasheer, weighed in air -	1,23
The same pieces when thoroughly soaked in water, weighed in air -	2,54
The same weighed in water at the temperature of 52° - -	0,72

Hence the specific gravity of the parcel when dry, is 2,412
And the specific gravity of the parcel when wet, is 1,396

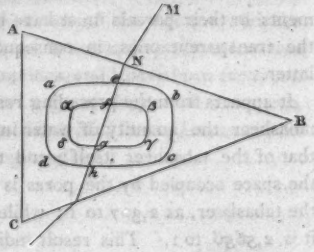
Mr. MACIE found the specific gravity of a parcel of opaque and transparent tabasheer to be 2,188; and Mr. CAVENDISH, having tried the same parcel, found it to be 2,169. The mean of Mr. JARDINE's results is 2,235, which exceeds the measures of MACIE and CAVENDISH, because the opaque fragments in their parcels must have been more numerous than the transparent ones, in consequence of the rarity of the latter.

It appears from the preceding results, that in both kinds of tabasheer the quantity of water imbibed exceeds in weight that of the tabasheer itself; and that in the opaque kinds, the space occupied by the pores is to the space occupied by the tabasheer, as 2,307 to 1; while in the transparent kinds, it is 2,5656 to 1. This result indicates a very remarkable

degree of porosity; and as it makes the pore more extensive in the transparent than in the opaque kind, contrary to what we should expect from their specific gravities, it seems to follow, that the water was not capable of insinuating itself into all the pores of the opaque tabasheer. This conclusion is rendered more probable, when we consider the extreme difficulty with which the oil of beech nut displaces the last portions of included air; and it affords a very plausible explanation of the fact, that the chalky tabasheer cannot be rendered transparent by the absorption of water.

We are now prepared by the preceding observations, for investigating the cause of the remarkable paradox exhibited by the transparent tabasheer, in becoming perfectly opaque and white, by absorbing a small quantity of water, and perfectly transparent when that quantity is increased. As this effect takes place indiscriminately with all fluids, it cannot be the result of any chemical action, and therefore its cause must be sought for in the changes which the light suffers in traversing the vacuities of the tabasheer.

Let ABC be a prism of this substance, and *abcd* one of its pores highly magnified. We know that this pore is filled with air; and that when a ray of light MN enters the separating surface *ab* at *e*, and quits it at *h*, it suffers so little refraction, and is therefore so little scattered, that the tabasheer appears transparent, and allows us to see objects



distinctly through it. This effect, which could not take place in any other porous substance, arises from the small difference between the refractive power of air and tabasheer. Let us now suppose that a small quantity of water is introduced into the pore *abcd*, so as not to fill it, but merely to line its circumference with a film contained between *abcd* and $\alpha\beta\gamma\delta$. Then the light which was formerly scattered by the slight refraction at *e* and *h* in passing from tabasheer into air, will now be a little less scattered at these points, since it passes from tabasheer into water, where the difference of refractive power is less; but in passing from the film of water into the air at *f*, and in entering the water again at *g*, the scattering of the rays will be very considerable, from the great difference in the refractive powers of air and water. In passing through every pore, therefore, the light is refracted, and consequently scattered no less than four times; and hence the piece of tabasheer must appear to be opaque. If we now saturate it with water, the pore *abcd* will be completely filled; the two great refractions which took place at *f* and *g*, will no longer exist; and the light will suffer only a slight refraction at *e* and *h*, by which it will be less scattered than when the tabasheer was dry. Hence it follows, that when the tabasheer is saturated with water, it ought to transmit the incident light freely, and to be more transparent than when it is quite dry, a result which is perfectly conformable to observation.

The very singular anomaly presented by the chalky tabasheer in becoming transparent by means of oil of beech nut, and not by means of oil of cassia or water, and the additional transparency communicated to the transparent

fragments by immersion in oil, if they do not arise from the existence of minute pores, which admit the oil and keep out the water, must be ascribed to a very considerable refractive density in the solid parts of the substance. The experiments made with the oil at different temperatures, indeed, seem to prove that the refractive power of the solid parts of the tabasheer is equal to about 1,500, or that of oil of beech nut raised to the temperature which produces the maximum transparency; but as the refractive power of a prism of tabasheer is greatly inferior to that of water, we shall now proceed to consider how these apparently opposite results can be reconciled, and what inferences can be drawn from them.

When alcohol is poured into water in a glass vessel, a scattering of the transmitted light is immediately visible, in consequence of their imperfect mixture, and of the difference of their refractive powers; but in a short time the union of the fluids is so complete, that the light is transmitted as freely as through either of them separately. Chemistry does not inform us how the particles of alcohol and of water are combined so as to produce this effect; but we know that the refractive power of the compound is intermediate between that of the two ingredients; and hence it is certain, that the refraction of the incident light is produced by their joint action. If we increase the quantity of water successively, the particles of alcohol will be thrown to a greater and greater distance, and the refractive power of the compound will be proportionally diminished. Let us now suppose that all the aqueous particles are extinguished, and that their place is occupied with a less refractive fluid, such as air; then it is obvious that the

refractive power of the compound will be inferior to that of water, and will approach to that of air, in proportion to the quantity of air united with the alcohol.

This hypothetical combination of air and alcohol, represents exactly what I conceive to be the condition of the dry and transparent tabasheer. The refractive power of the solid parts is not far from 1,500; but the substance is so formed, that its particles are kept at a great distance by means of vacuities filled with air, and arranged in such a manner, that the light passes from the particles of tabasheer into the particles of air without suffering refraction, in the same manner as it passes from the particles of alcohol to the particles of water, when these fluids are chemically united. That there is not a chemical union between the air and the tabasheer, is certain, because the air may be displaced, and the transparency preserved in *vacuo*; and hence we may draw the important inference, of which we have no other example in physics, that the tabasheer and its included air exercise a joint action upon light, in the same manner as if they were in a state of chemical union. I have the honour to be, &c. &c. &c.

DAVID BREWSTER.

Edinburgh, March 2, 1819.

XX. *An Account of a Membrane in the Eye, now first described.* By Arthur Jacob, *M. D. Member of the Royal College of Surgeons in Ireland, Demonstrator of Anatomy and Lecturer on Diseases of the Eye in the University of Dublin. Communicated by* James Macartney, *M. D. F. R. S.*

Read July 1, 1819.

ANATOMISTS describe the retina as consisting of two portions, the medullary expansion of the nerve, and a membranous or vascular layer. The former externally, next to the choroid coat, and the latter internally, next to the vitreous humor.* All however, except ALBINUS and some of his disciples, agree, that the nervous layer cannot be separated so as to present the appearance of a distinct membrane, though it may be scraped off, leaving the vascular layer perfect. That the medullary expansion of the optic nerve is supported by a vascular layer, does not I think admit of doubt; but it does not appear that ALBINUS was right in supposing that the nervous layer can be separated in form of a distinct membrane, though shreds of a considerable size may be detached, especially if hardened by acid or spirit.

Exclusive of these two layers, I find that the retina is covered on its external surface by a delicate transparent

* RUYSCH. Epist. Anat. Prob. xiii. ALBINUS, Annot. Acad. lib. iii. cap. xiv. HALLER, Elem. Phys. T. v. lib. xvi. sect. 2. ZINN. Descrip. Anat. Oculi. cap. iii. sect. iii. SABATIER, BOYER, CHARLES BELL, CUVIER, &c. &c.

membrane, united to it by cellular substance and vessels. This structure, not hitherto noticed by anatomists, I first observed in the spring of the last year, and have since so frequently demonstrated, as to leave no doubt on my mind of its existence as a distinct and perfect membrane, apparently of the same nature as that which lines serous cavities. I cannot describe it better, than by detailing the method to be adopted for examining and displaying it. Having procured a human eye, within forty-eight hours after death, a thread should be passed through the layers of the cornea, by which the eye may be secured under water, by attaching it to a piece of wax, previously fastened to the bottom of the vessel, the posterior half of the sclerotic having been first removed. With a pair of dissecting forceps in each hand, the choroid coat should be gently torn open and turned down. If the exposed surface be now carefully examined, an experienced eye may perceive, that this is not the appearance usually presented by the retina; instead of the blue-white reticulated surface of that membrane, a uniform villous structure, more or less tinged by the black pigment, presents itself. If the extremity of the ivory handle of a dissecting knife be pushed against this surface, a breach is made in it, and a membrane of great delicacy may be separated and turned down in folds over the choroid coat, presenting the most beautiful specimen of a delicate tissue which the human body affords. If a small opening be made in the membrane, and the blunt end of a probe introduced beneath, it may be separated throughout, without being turned down, remaining loose over the retina; in which state if a small particle of paper or globule of air be introduced under it, it is raised so as to be seen against the

light, and is thus displayed to great advantage; or it is sometimes so strong as to support small globules of quicksilver dropped between it and the retina, which renders its membranous nature still more evident. If a few drops of acid be added to the water after the membrane has been separated, it becomes opaque and much firmer, and may thus be preserved for several days, even without being immersed in spirit.

That it is not the nervous layer which I detach, is proved by the most superficial examination; first, because it is impossible to separate that part of the retina, so as to present the appearance I mention;* and, secondly, because I leave the retina uninjured, and presenting the appearance described by anatomists, especially the yellow spot of SOEMMERRING, which is never seen to advantage until this membrane be removed: and hence it is that that conformation, as well as the fibrous structure of the retina in some animals, becomes better marked from remaining some time in water, by which the membrane I speak of is detached.

The extent and connections of this membrane are sufficiently explained by saying, that it covers the retina from the optic nerve to the ciliary processes. To enter into farther investigation on this subject, would lead to a discussion respecting the structure of the optic nerve, and the termination of the retina anteriorly, to which it is my intention to return at a future period.

The appearance of this part I find to vary in the different classes of animals and in man, according to age and other circumstances. In the fœtus of nine months it is exceed-

* See HALLER, ZINN, &c. loc. cit.

ingly delicate, and with difficulty displayed. In youth it is transparent, and scarcely tinged by the black pigment. In the adult it is firmer, and more deeply stained by the pigment, which sometimes adheres to it so closely as to colour it almost as deeply as the choroid coat itself; and to those who have seen it in this state, it must appear extraordinary that it should not have been before observed. In one subject, aged fifty, it possessed so great a degree of strength as to allow me to pass a probe under it, and thus convey the vitreous humor covered by it and the retina from one side of the basin to the other; and in a younger subject I have seen it partially separated from the retina by an effused fluid. In the sheep, ox, horse, or any other individual of the class mammalia which I have had an opportunity of examining, it presents the same character as in man; but is not so much tinged by the black pigment, adheres more firmly to the retina, is more uniform in its structure, and presents a more elegant appearance when turned down over the black choroid coat. In the bird, it presents a rich yellow brown tint, and when raised, the blue retina presents itself beneath; in animals of this class, however, it is difficult to separate it to any extent, though I can detach it in small portions. In fishes, the structure of this membrane is peculiar and curious. It has been already described as the medullary layer of the retina by HALLER and CUVIER,* but I think incorrectly, as it does not present any of the characters of nervous structure, and the retina is found perfect beneath it. If the sclerotic coat be removed behind, with the choroid coat and gland so

* Element. Phys. T. v. lib. xvi. sect. ii. CUVIER Leçons d'Anat. Comp. T. ii. p. 419.

called, the black pigment is found resting upon, and attached to, a soft friable thick fleecy structure, which can only be detached in small portions, as it breaks when turned down in large quantity. Or if the cornea and iris be removed anteriorly, and the vitreous humor and lens withdrawn, the retina may be pulled from the membrane, which remains attached to the choroid coat, its inner surface not tinged by the black pigment, but presenting a clear white, not unaptly compared by HALLER to snow.

Besides being connected to the retina, I find that the membrane is also attached to the choroid coat, apparently by fine cellular substance and vessels; but its connection with the retina being stronger, it generally remains attached to that membrane, though small portions are sometimes pulled off with the choroid coat. From this fact I think it follows, that the accounts hitherto given of the anatomy of these parts, are incorrect. The best anatomists* describe the external surface of the retina as being merely in contact with the choroid coat, as the internal with the vitreous humor, but both totally unconnected by cellular membrane, or vessels, and even having a fluid secreted between them: some indeed speak loosely and generally of vessels passing from the choroid to the retina; but obviously not from actual observation, as I believe no one has ever seen vessels passing from the one membrane to the other. My observations lead me to conclude, that wherever the different parts of the eye are in

* See HALLER, Elem. Phys. T. v. lib. xvi. sect. ii. ZINN. cap. ii. sect. i. § ii. BOYER, Anat. T. iv. p. 113. SABATIER, T. ii. p. 70. BICHAT, Anat. Descr. T. ii. p. 447. CUVIER, Leçons d'Anat. Comp. T. ii. p. 418. CHARLES BELL, Anat. vol. iii. p. 51. RIBES, Mem. de la Soc. Med. d'Emulation, T. viii. p. 633.

contact, they are connected to each other by cellular substance, and, consequently, by vessels; for I consider the failure of injections no proof of the want of vascularity in transparent and delicate parts, though some anatomists lay it down as a criterion. Undoubtedly the connection between these parts is exceedingly delicate, and, hence, is destroyed by the common method of examining this organ; but I think it is proved in the following way. I have before me the eye of a sheep killed this day, the cornea secured to a piece of wax fastened under water, and the posterior half of the sclerotic coat carefully removed. I thrust the point of the blade of a pair of sharp scissors through the choroid coat into the vitreous humor, to the depth of about an eighth of an inch, and divide all, so as to insulate a square portion of each membrane, leaving the edges free, and consequently no connection except by surface; yet the choroid does not recede from the membrane I describe, the membrane from the retina, nor the retina from the vitreous humor. I take the end of the portion of choroid in the forceps, turn it half down, and pass a pin through the edge, the weight of which is insufficient to pull it from its connection. I separate the membrane in like manner, but the retina I can scarcely detach from the vitreous humor, so strong is the connection. The same fact may be ascertained by making a transverse vertical section of the eye, removing the vitreous humor from the posterior segment, and taking the retina in the forceps, pulling it gently from the choroid, when it will appear beyond a doubt that there is a connection between them.

Let us contrast this account of the matter with the common one. The retina, a membrane of such delicacy, is

described as being extended between the vitreous humor and choroid, from the optic nerve to the ciliary processes, being merely laid between them, without any connection, and the medullary fibres in contact with a coloured mucus retained in its situation by its consistence alone. This account is totally at variance with the general laws of the animal economy: in no instance have we parts, so dissimilar in nature, in actual contact: wherever contact without connection exists, each surface is covered by a membrane, from which a fluid is secreted; and wherever parts are united, it is by the medium of cellular membrane, of which serous membrane may be considered as a modification. If the retina be merely in contact with the vitreous humor and choroid, we argue from analogy, that a cavity lined by serous membrane exists both on its internal and external surface; but this is not the fact. In the eye a distinction of parts was necessary, but to accomplish this a serous membrane was not required; it is only demanded where great precision in the motion of parts was indispensable, as in the head, thorax, and abdomen; a single membrane, with the interposition of cellular substance, answers the purpose here. By this explanation we surmount another difficulty, the unphilosophical idea of the colouring matter being laid on the choroid, and retained in its situation by its viscidity, is discarded; as it follows, if this account be correct, that it is secreted into the interstices of fine cellular membrane here, as it is upon the ciliary processes, back of the iris, and pecten, under the conjunctiva, round the cornea, and in the edge of the membrana nictitans and sheath of the optic nerve in many animals. Dissections are recorded where fluids have been found collected between the choroid and

retina, by which the structure of the latter membrane was destroyed; the explanation here given is as sufficient to account for the existence of this fluid, as that which attributes it to the increased secretion of a serous membrane.

I take this opportunity of describing the method I adopt for examining and displaying these and other delicate parts, a method, which though simple, will, I expect, prove an important improvement in the means of scrutinizing the structure of animal and vegetable bodies. I procure a hollow sphere of glass from two to three inches in diameter, about one fourth of which is cut off at the part where it is open, and the edges ground down, so as to fit accurately upon a piece of plate glass, the surface of which is also ground; the object to be examined is attached to a piece of wax fastened upon the plate of glass and immersed in a basin of water, with the cut sphere, which is inverted over it, of course full of water, and the whole withdrawn from the basin. The part may thus be examined under the most favourable circumstances; it floats in water, the only method by which delicate parts can be unfolded and displayed: the globular form of the vessel answers the purpose of a lens of considerable power and perfection, at the same time that it admits light in any quantity or direction to illuminate the object; and, what is of the utmost importance, a preparation of the greatest delicacy may thus be handed round a class in safety.

XXI. *A new method of solving numerical equations of all orders, by continuous approximation.** *By* W. G. Horner, *Esq. Communicated by* Davies Gilbert, *Esq. F. R. S.*

Read July 1, 1819.

1. THE process which it is the object of this Essay to establish, being nothing else than the leading theorem in the Calculus of Derivations, presented under a new aspect, may be regarded as a universal instrument of calculation, extending to the composition as well as analysis of functions of every kind. But it comes into most useful application in the numerical solution of equations.

2. ARBOGAST's developement of

$$\phi(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \ldots\ldots)$$

(See *Calc. des Der.* § 33) supposes all the coefficients within the parenthesis to be known previously to the operation of ϕ. To the important cases in which the discovery of γ, δ, &c. depends on the previous developement of the partial functions

$$\phi(\alpha + \beta x), \phi(\alpha + \beta x + \gamma x^2), \text{ \&c.}$$

* The only object proposed by the author in offering this Essay to the acceptance of the Royal Society, for admission into the Philosophical Transactions, is to secure beyond the hazard of controversy, an Englishman's property in a useful discovery. Useful he may certainly be allowed to call it, though the produce of a purely mathematical speculation; for of all the investigations of pure mathematics, the subject of *approximation* is that which comes most directly, and most frequently into contact with the practical wants of the calculator.

How far the manner in which he has been so fortunate as to contemplate it has conduced, by the result, to satisfy those wants, it is not for him to determine; but his belief is, that both Arithmetic and Algebra have received some degree of improvement, and a more intimate union. The abruptness of transition has been broken down into a gentle and uniform acclivity.

it is totally inapplicable. A theorem which should meet this deficiency, without sacrificing the great facilitating principle of attaching the functional symbols to a alone, does not appear to have engaged the attention of mathematicians, in any degree proportionate to the utility of the research. This desideratum it has been my object to supply. The train of considerations pursued is sufficiently simple; and as they have been regulated by a particular regard to the genius of arithmetic, and have been carried to the utmost extent, the result seems to possess all the harmony and simplicity that can be desired; and to unite to continuity and perfect accuracy, a degree of facility superior even to that of the best popular methods.

Investigation of the Method.

3. In the general equation

$$\varphi x = o$$

I assume $x = \mathrm{R} + r + r' + r'' + \ldots\ldots$

and preserve the binomial and continuous character of the operations, by making successively

$$\begin{aligned} x &= \mathrm{R} + z = \mathrm{R} + r + z' \\ &= \mathrm{R}' + z' = \mathrm{R}' + r' + z'' \\ &= \mathrm{R}'' + z'' = \&c. \end{aligned}$$

Where R^s represents the whole portion of x which has already been subjected to φ, and $z_s = r^s + z^s$ the portion still excluded; but of which the part r^s is immediately ready for use, and is to be transferred from the side of z to that of R, so as to change $\varphi \mathrm{R}^s$ to $\varphi \mathrm{R}^{s'}$ without suspending the corrective process.

4. By TAYLOR's theorem, expressed in the more convenient manner of ARBOGAST, we have

$$\varphi x = \varphi(\mathrm{R} + z) =$$
$$\varphi \mathrm{R} + \mathrm{D}\varphi \mathrm{R} \,.\, z + \mathrm{D}^2 \varphi \mathrm{R} \,.\, z^2 + \mathrm{D}^3 \varphi \mathrm{R} \,.\, z^3 + \ldots\ldots$$

Where by $D^n \varphi R$ is to be understood $\frac{d^n \varphi R}{1 \cdot 2 \ldots \ldots n \cdot dR^n}$, viz. the n^{th} derivee with its proper denominator; or, that function which ARBOGAST calls the *derivée divisée*, and distinguishes by a *c* subscribed. Having no occasion to refer to any other form of the derivative functions, I drop the distinctive symbol for the sake of convenience. Occasionally these derivees will be represented by *a*, *b*, *c*, &c.

5. Supposing φR and its derivees to be known, the mode of valuing $\varphi R'$ or $\varphi (R + r)$ is obvious. We have only to say in the manner of LAGRANGE, when preparing to develope his Theory of Functions,

$$\begin{aligned}
\varphi R' &= \varphi R + Ar \\
A &= D\varphi R + Br \\
B &= D^2\varphi R + Cr \\
C &= D^3\varphi R + Dr \\
&\ldots\ldots\ldots \\
V &= D^{n-1}\varphi R + Ur \\
U &= D^{n-1}\varphi R + r \qquad \ldots\ldots [1.]
\end{aligned}$$

Taking these operations in reverse order, we ascend with rapidity to the value of $\varphi (R + r)$ or $\varphi R'$.

6. The next point is, to apply a similar principle to discover the value of $\varphi (R + r + r') = \varphi (R' + r') = \varphi R''$. We here have

$$\begin{aligned}
\varphi R'' &= \varphi R' + A'r' \\
A' &= D\varphi R' + B'r' \\
B' &= D^2\varphi R' + C'r' \\
C' &= D^3\varphi R' + D'r' \\
&\ldots\ldots\ldots \\
V' &= D^{n-1}\varphi R' + U'r' \\
U' &= D^{n-1}\varphi R' + r'
\end{aligned}$$

But the former operation determined $\varphi R'$ only, without giving the value of any of the derived functions. The very simple

scale of known quantities, therefore, by which we advance so rapidly in the first process, fails in those which follow.

7. Still we can reduce these formulæ to known terms; for since we have in general

$$D^r\, D'^s \varphi\alpha = \frac{r+1 \,.\, r+2 \,.\,\ldots\ldots\, r+s}{1 \quad 2 \qquad\qquad\quad s} D^{r+s} \varphi\alpha$$

(See ARBOGAST, § 137); by applying a similar reduction to the successive terms in the developement of $D^m \varphi R' = D^m \varphi(R+r)$, we obtain*

$$D^m \varphi R' = D^m \varphi R + \frac{m+1}{1} D^{m+1} \varphi R \,.\, r + \frac{m+1}{1} \,.\, \frac{m+2}{2} D^{m+2} \varphi R \,.\, r^2 + \frac{m+1 \,.\,\ldots\ldots\, m+3}{1 \qquad\qquad\quad 3} D^{m+3} \varphi R \,.\, r^3 + \&c.$$

And it is manifest that this expression may be reduced to a form somewhat more simple, and at the same time be accommodated to our principle of successive derivation, by introducing the letters A, B, C, &c. instead of the functional expressions.

8. As a general example, let

$$M = D^m \varphi R + Nr$$
$$N = D^{m+1} \varphi R + Pr$$
$$P = D^{m+2} \varphi R + Qr$$
$$\ldots\ldots\ldots\ldots$$

* This theorem, of which that in Art. 4 is a particular case [$m = o$], has been long in use under a more or less restricted enunciation, in aid of the transformation of equations. HALLEY's *Specimen Analyticum*, NEWTON's limiting equations, and the formulæ in SIMPSON's Algebra (ed. 5, p. 166, *circa fin.*) are instances. In a form still more circumscribed [$r=1$, $R=o$, 1, 2, &c.] it constitutes the *Nouvelle Methode* of BUDAN; which has been deservedly characterized by LAGRANGE as simple and elegant. To a purpose which will be noticed hereafter, it applies very happily; but regarded as an instrument of approximation, its extremely slow operation renders it perfectly nugatory: and as LEGENDRE justly reported, and these remarks prove, it has not the merit of originality.

represent any successive steps in the series in Art. 5; then are

$$D^{m} \quad \varphi R = M - Nr$$
$$D^{m+1}\varphi R = N - Pr$$
$$D^{m+2}\varphi R = P - Qr$$
$$\ldots\ldots\ldots\ldots$$

And by substituting these equivalents in the developement just enounced, it becomes

$$D^{m}\varphi R' = M + m\,Nr + \frac{m \cdot m+1}{1\ \ 2}Pr^{2} + \frac{m \ldots m+2}{1 \ldots\ 3}Qr^{3} + \&c.$$

9. With this advantage, we may now return to the process of Art. 6, which becomes

$$\varphi R'' = \varphi R' + A'r'$$
$$A' = (A + Br + Cr^{2} + Dr^{3} + Er^{4} + \&c.) + B'r'$$
$$B' = (B + 2\,Cr + 3\,Dr^{2} + 4\,Er^{3} + \&c.) + C'r'$$
$$C' = (C + 3\,Dr + 6\,Er^{2} + \&c.) + D'r'$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$V' = (V + \tfrac{n-2}{1}Ur + \tfrac{n-2 \cdot n-1}{1\ \cdot\ 2}r^{2}) + U'r'$$
$$U' = (U + \tfrac{n-1}{1}r) + r' \qquad \ldots\ldots [\text{II.}]$$

Taking these operations in reverse order as before, by determining U′, V′ .... C′, B′, A′, we ascend to the value of $\varphi R''$.

10. In this theorem, the principle of successive derivation already discovers all its efficacy; for it is obvious that the next functions U″, V″ ..... C″, B″, A″, $\varphi R''$, flow from the substitution of A′, B′, C′, ... V′, U′, $\varphi R''$, r', r'', for A, B, C .... V, U, $\varphi R'$, r, r', in these formulæ; and from these U‴, V‴, &c.; and so on to any desirable extent. In this respect, Theorem II, algebraically considered, perfectly answers the end proposed in Art. 2.

11. We perceive also, that some advance has been made toward arithmetical facility; for all the figurate coefficients

here employed are lower by one order than those which naturally occur in transforming an equation of the n^{th} degree. But it is much to be wished, that these coefficients could be entirely dispensed with. Were this object effected, no multipliers would remain, except the successive corrections of the root, and the operations would thus arrange themselves, in point of simplicity, in the same class as those of division and the square root.

12. Nor will this end appear unattainable, if we recur to the known properties of figurate numbers; which present to our view, as *equivalent to the* n^{th} *term of the* m^{th} *series:*

1. *The difference of the* n^{th} *and* $n-1^{th}$ *term of the* $m+1^{th}$ *series.*

2. *The sum of the first n terms of the* $m-1^{th}$ *series.*

3. *The sum of the* n^{th} *term of the* $m-1^{th}$, *and the* $n-1^{th}$ *term of the* m^{th} *series.*

The depression already attained has resulted from the first of these properties, and a slight effort of reflection will convince us that the second may immediately be called to our aid.

13. For this purpose, let the results of Art. 9 be expressed by the following notation:

$$\begin{aligned}
\phi R'' &= \phi R' + A' r' \\
A' &= A_1 + B' r' \\
B' &= B_2 + C' r' \\
C' &= C_3 + D' r' \\
&\cdots\cdots\cdots \\
V' &= V_{n-2} + U' r' \\
U' &= U_{n-1} + r'
\end{aligned}$$

the exponents subjoined to any letter indicating the degree

of the figurate coefficients in that formula of the theorem, of which such letter is the first term.

14. Although this statement appears only to have returned to us the conditions of Art. 6, with all their disadvantages, and to have merely substituted

$$A_1 \text{ for } D\phi R' \text{ or } a'$$
$$B_2 \text{ for } D^2\phi R' \text{ or } b'$$
$$C_3 \text{ for } D^3\phi R' \text{ or } c'$$

&c. yet, by means of the property just alluded to, the essential data A, B, C, &c. which have disappeared, will again be extricated. For the developement of $D^m\phi R'$, found in Art. 8, undergoes thereby the following analysis:

$$\begin{aligned} M + m\,Nr + \tfrac{m\,.\,m+1}{1\,.\,2}Pr^2 + \tfrac{m\,.\,m+1\,.\,m+2}{1\,.\,2\,.\,3}Qr^3 + \ldots\ldots \\ = M + Nr + \quad Pr^2 + \quad Qr^3 + \ldots\ldots \\ + Nr + 2\,Pr^2 + 3\,Qr^3 + \ldots\ldots \\ + Nr + 3\,Pr^2 + 6\,Qr^3 + \ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + Nr + m\,Pr^2 + \tfrac{m\,.\,m+1}{1\,.\,2}Qr^3 + \ldots\ldots \end{aligned}$$

which equivalence will be thus expressed:

$$M_m = M + N_1 r + N_2 r + N_3 r + \ldots\ldots\ldots + N_m r$$

Returning therefore once more to our theorem, we now have

$$\begin{aligned} \phi R'' &= \phi R' + A' r' \\ A &= (A + B_1 r) + B' r' \\ B' &= (B + C_1 r + C_2 r) + C' r' \\ C' &= (C + D_1 r + D_2 r + D_3 r) + D' r' \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ V' &= (V + U_1 r + U_2 r + U_3 r + \ldots\ldots U_{n-1} r) + U' r' \\ U' &= (U + \overline{n-1}\,.\,r) + r' \end{aligned}$$

15. This theorem employs exactly the same total number of addends as Theorem II, but with the important improvement, that the number of addends to each derivee is inversely as their magnitude, contrary to what happened before. Figurate multipliers are also excluded. And it is easy to convince ourselves that no embarrassment will arise from the newly introduced functions. For if we expand any of the addends $N_k r$ in the general formula equivalent to M_m, and analyze it by means of the *third property* of figurate series, we shall find

$$M_k r = N_{k-1} r + P_k rr.$$

And since we take the scale in our Theorem in a reverse or ascending order, this formula merely instructs us to multiply an addend already determined by r, and to add the product to another known addend; and if we trace its effect through all the descending scale, to the first operations, we observe that the addends to the last derivee, from which the work begins, are simply r repeated $n-1$ times.

16. Because $N_o = N$, the addend exterior to the parenthesis, might for the sake of uniformity be written $N_o' r'$. The harmony of the whole scheme would then be more completely displayed. To render the simplicity of it equally perfect, we may reflect that as the factors r, r', &c. are engaged in no other manner than has just been stated, viz. in effecting the subordinate derivations, their appearance among the principal ones is superfluous, and tends to create embarrassment. Assume therefore

$$_kN = N_k r,$$

and we have

$$\varphi R'' = \varphi R' + {}_{0}A'$$
$$A' = (A + {}_{1}B) + {}_{0}B'$$
$$B' = (B + {}_{1}C + {}_{2}C) + {}_{0}C'$$
$$C' = (C + {}_{1}D + {}_{2}D + {}_{3}D) + {}_{0}D'$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$V' = (V + {}_{1}U + {}_{2}U + {}_{3}U + \ldots\ldots {}_{n-2}U) + {}_{0}U'$$
$$U' = (U + \overline{n-1} \cdot r) + r' \qquad [III]$$

the subordinate derivations being understood.

17. The Theorems hitherto give only the synthesis of φx, when $x = R + r + r' +$ &c. is known. To adapt them to the inverse or analytical process, we have only to subtract each side of the first equation from the value of φx; then assuming $\varphi x - \varphi R^{s} = \Delta^{s}$, we have

$$\Delta' = \Delta - {}_{0}A$$
$$A = a + {}_{0}B$$

&c. as in Theorem I.

$$\Delta'' = \Delta' - {}_{0}A'$$
$$A' = (A + {}_{1}B) + {}_{0}B'$$

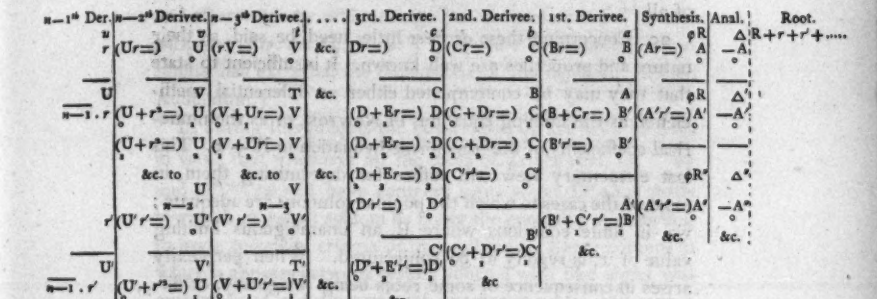
&c. as in Theorem II. or III.

The successive invention of R, r, r', &c. will be explained among the numerical details. In the mean time, let it be observed that these results equally apply to the popular formula $\varphi x =$ constant, as to $\varphi x = 0$.

18. I shall close this investigation, by exhibiting the whole chain of derivation in a tabular form. The calculator will then perceive, that the algebraic composition of the addends no longer requires his attention. He is at liberty to regard the characters by which they are represented, in the light of mere corresponding symbols, whose origin is fully explained

at their first occurrence in the table, and their ultimate application at the second. The operations included in the parentheses may be mentally effected, whenever r is a simple digit. And lastly, the vertical arrangement of the addends adapts them at once to the purposes of arithmetic, on every scale of notation.

General Synopsis.



Illustrations.

19. The remarks which are yet to be adduced will bear almost exclusively on the Analytic portion of the Theorem, from which the Synthetic differs only in the less intricate management of the first derivee; this function having no concern with the discovery of the root, and its multiple being additive like all the rest, instead of subtractive.

From the unrestricted nature of the notation employed, it is evident that no class of equations, whether finite, irrational or transcendental, is excluded from our design. In this respect indeed, the new method agrees with the established

popular methods of approximation; a circumstance in favour of the latter, which is overlooked by many algebraists, both in employing those methods, and in comparing them with processes pretending to superior accuracy. The radical feature which distinguishes them from ours is this: they forego the influence of all the derivees, excepting the first and perhaps the second; ours provides for the effectual action of all.

20. Concerning these *derivées* little need be said, as their nature and properties are well known. It is sufficient to state that they may be contemplated either as differential coefficients, as the limiting equations of NEWTON, or as the numerical coefficients of the transformed equation in $R + z$. This last elementary view will suffice for determining them, in most of the cases to which the popular solutions are adequate; viz. in finite equations where R, an unambiguous limiting value of x, is readily to be conjectured. When perplexity arises in consequence of some roots being imaginary, or differing by small quantities, the second notation must be called in aid. The first, in general, when ϕx is irrational or transcendental.

21. The fact just stated, namely, that our theorem contains within itself the requisite conditions for investigating the limits, or presumptive impossibility, of the roots, demonstrates its sufficiency for effecting the developement of the real roots, independently of any previous knowledge of R. For this purpose, we might assume $R = o; r, r$, &c. $= 1$ or $.1$ &c. and adopt, as most suitable to these conditions, the algorithm of Theorem II, until we had arrived at R^{x}, an unambiguous limiting value of x. But since these initiatory researches

seem more naturally to depend on the simple derivees, a, b, &c. than on A, B, &c. their aggregates; and since, in fact, as long as r is assumptive or independent of R, our system of derivation offers no peculiar advantage; I should prefer applying the limiting formulæ in the usual way; passing however from column to column (Wood, § 318.) of the results, at first by means of the neat algorithm suggested in the note on Art. 7, and afterwards by differencing, &c. as recommended by Lagrange, (*Res. des Eq. Num.* § 13), when the number of columns has exceeded the dimensions of the equation. (Vide Addendum.)

If, during this process the observation of De Gua be kept in view, that whenever all the roots of φx are real, $D^{m-1}\varphi x$ and $D^{m+1}\varphi x$ will have contrary signs when $D^m\varphi x$ is made to vanish, we shall seldom be under the necessity of resorting to more recondite criteria of impossibility. Every column in which o appears between results affected with like signs, will apprize us of a distinct pair of imaginary roots; and even a horizontal change of signs, occurring between two horizontal permanences of an identical sign, will induce a suspicion, which it will in general be easy, in regard of the existing case, either to confirm or to overthrow.

22. The facilities here brought into a focus, constitute, I believe, a perfectly novel combination; and which, on that account, as well as on account of its natural affinity to our own principles, and still more on account of the extreme degree of simplicity it confers on the practical investigation of limits, appears to merit the illustration of one or two familiar examples.

Ex. 1. Has the equation $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0$ any real root?—See EULER, C. D. p. 678.

$x =$	0	1	2
	20	9	4
	−16	−8	0
	8	2	8
	−4	0	4
	1	1	1

Here the first column consists of the given co-efficients taken in reverse order. In the second, 9 is = the sum of the first column, -8 is $= -16 + 2\,(8) + 3\,(-4) + 4\,(1)$, 2 is $= 8 + 3\,(-4) + 6\,(1)$, &c. The third column is formed from the second, by the same easy process. We need proceed no farther; for the sequences 2, 0, 1 in the second column, and 4, 0, 8 in the third, show that the equation has two pairs of imaginary roots. Consequently it has no real root.

Ex. 2. To determine the nearest distinct limits of the positive roots of $x^3 - 7x + 7 = 0$. See LAGRANGE, *Res. des E. N.* § 27, and note 4. § 8.

Operating as in the former example, we have

$x =$	0	1	2
	7	1	1
	−7	−4	5
	0	3	6
	1	1	1

Since all the signs are now positive, 2 is greater than any of the positive roots. Again, between -4 and $+5$, it is manifest, that 0 will occur as a value of the first derivee, and

that the simultaneous value of the second derivee will be affirmative. But as the principal result has evidently converged and subsequently diverged again in this interval, no conclusion relative to the simultaneous sign of that result can be immediately drawn. We will return to complete the transformations.

For $x=$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
	1000	631	328	97	−56	−125	−104	18
	−400	−337	−268					
	30	33	36					
	1	1	1					

Here the first column was formed from that under $x=1$, by annexing ciphers according to the dimensions of the functions; the 2nd and 3rd columns and the number 97 were found as in the former Example; the remaining numbers by differencing and extending the series 1000,631,328,97. We have no need to continue the work, since the changes of signs in the principal results indicate the first digits of the roots in question to be 1.3 and 1.6. But if we proceed by farther differencing to complete all the lines, the columns standing under these numbers will give the co-efficients of $\phi(1.3+z)$ and $\phi(1.6+z)$ without farther trouble.

23. Assuming, then, that R has been determined, and $R+z$ substituted for x in the proposed equation, thereby transforming it to

$$\Delta = az + bz^2 + cz^3 + dz^4 + \ldots\ldots\ldots$$

it is to this latter equation that the analytical part of our theorem is more immediately adapted. Now the slightest degree of reflection will evince, that our method is absolutely identical for all equations of the same order, whether they

be binomial or adfected, as soon as the transformation in R has been accomplished. The following description, therefore, of a familiar process in arithmetic, will convey an accurate general idea of our more extensive calculus, and obviate the necessity of any formal precepts.

In EVOLUTION, the first step is unique, and if not assisted by an effort of memory, could only be tentative. The whole subsequent process may be defined, *division by a variable divisor.* For an accurate illustration of this idea, as discoverable in the existing practice of arithmeticians, we cannot however refer to the mode of extracting any root, except that of the square; and to this, only in its most recently improved state. Here, in passing from one divisor to another, *two additive corrections* are introduced; the first depending on the last correction of the root, the second on the correction actually making. And this *new quotient correction* of the root, since it must exist previously to the completion of the divisor by which it is to be verified, *is required to be found by means of the incomplete divisor*; and may be taken out, either to one digit only, as is most usual, or to a number of digits equal to that which the complete and incomplete divisors possess in common. And farther, as these *divisors* may not, *in the first* instance, agree accurately even in a single digit, it is necessary at that stage of the operation, mentally to anticipate the effect of the new quotient, so as to obtain a sufficiently correct idea of the magnitude of the new divisor.

24. This is an accurate statement of the relation which the column headed by the first derivee bears to the analysis. The remaining columns contribute their aid, as successively subsidiary to each other; the contributions commencing with

the last or $n-1^{th}$ derivee, and being conveyed to the first through a regular system of *preparatory addends* dependent on the last quotient-correction, and of *closing addends* dependent on the new one. The *overt and registered* manner of conducting the whole calculation, enables us to derive important advantage from *anticipated corrections* of the divisors, not only at the first step, but, if requisite, through the whole performance, and also, without the necessity of a minute's bye-calculation, communicates, with the result, its *verification*.

25. Let us trace the operation of the theorem as far as may be requisite, through the ascending scale of equations.

1. In *Simple equations*, the reduced equation may be represented by $\Delta = ax$; whence $x = \frac{\Delta}{a}$. Now the theorem directs us to proceed thus:

$$\begin{array}{ll} a & \Delta\,(r + r' + \ldots \\ & \frac{-ar}{\Delta'} \\ & \frac{-ar'}{\Delta''} \\ & \frac{-ar'}{\Delta'''} \\ & \&c. \end{array}$$

precisely the common arithmetical process of division.

2. In *Quadratics*, we have $\Delta = ax + x^2$, and proceed in this manner:

$$\begin{array}{lll} 1 & a & \Delta\,(r + r' + \ldots \\ & \frac{r}{A} & \frac{-Ar}{\Delta'} \\ & r & \frac{-A'r'}{\Delta''} \\ & \frac{r'}{A'} & \&c. \\ & \&c. & \end{array}$$

the known arithmetical process for extracting the square root.

3. At *Cubic equations*, the aberration of the old practice of evolution commences, and our theorem places us at once on new ground. We have here

$$\Delta = ax + bx^4 + x^9$$

and must proceed thus:

1	b	a	$\Delta(r+r+\dots$
	r	$B=_0 B$	$-Ar$
	B	A	Δ'
	$2r$	$B+r=_1 B$	$-A'r'$
	r'	$B'r'$	Δ''
	B'	A'	&c.
	&c.	&c.	

This *ought to be* the arithmetical practice of the cube root, as an example will prove.

Ex. I. *Extract the cube root of* 48228544.

Having distributed the number into tridigital periods as usual, we immediately perceive that the first figure of the root is $3 = R$. Consequently, the first subtrahend is $R^3 = 27$, the first derivee $3R' = 27$, the second $3R = 9$; the third $(=1,)$ need not be written. Hence

			48228544(364
9.	27..		27
6	576		
96	3276	19656	21228
12.	612..		
4	4336		1572544
1084	393136	1572544	

In this example the reader will perceive that no supplementary operations are concealed. The work before him is complete, and may be verified mentally. I need not intimate

how much more concise it is than even the abbreviated statement of the old process. (See HUTTON's *Course.*)

The station of 1, 2, &c. numeral places respectively, which the closing addends occupy in advance of the preparatory ones, is an obvious consequence of combining the numeral relation of the successive root-figures with the potential relation of the successive derivees. In fact, as is usual in arithmetic, we tacitly regard the last root-figure as units, and the new one as a decimal fraction; then the common rules of decimal addition and multiplication regulate the vertical alineation of the addends.

26. The advantage of mental verification is common to the solution of equations of every order, provided the successive corrections of the root be simple digits: for the parenthetic derivations will, in that case, consist of multiplying a given number by a digit, and adding the successive digital products to the corresponding digits of another given number; all which may readily be done without writing a figure intermediate to these given numbers and the combined result. For this reason the procedure by single digits appears generally preferable.

Nevertheless, to assist the reader in forming his own option, and at the same time to institute a comparison with known methods on their own grounds, I introduce one example illustrative of the advantage which arises from the anticipatory correction of the divisors spoken of in Art. 24, when the object is to secure a high degree of convergency by as few approximations as possible. The example is that by which NEWTON elucidates his method. I premise as the depreciators of NEWTON do, that it is an extremely easy

problem; and I say this to invite comparison, not so much with his mode of treating it, as with theirs.

Ex. II. What is the value of x in the equation $x^3-2x=5$.

The root is manifestly a very little greater than 2. Make it $x=2+z$, and the equation becomes

$$1=10z+6z^2+z^3.$$

Hence, arranging the derivees,

6. 10. 1.000(
6

The first digit will obviously be so nearly 1, that by anticipating its effect on the divisor, we are sure this will be very nearly 106. Hence

10.6) 1.000(.094 first correction

The square is $94^2=8836$.

Hence we have

6... 10..... 1.000000000(.094
094 572836 993846584

6094×94= 10572836 6158416
188 581672
3

The first digit of the next correction will evidently be 5; the effect of which we have as before anticipated as far as one digit. The divisor will therefore be 11158 correct to the last figure. Hence

11158)6158416(55148, second correction.

The square is 30418, &c. to 10 digits.

Hence,

8094 10572836 6158416
18855148 581672

628255148×5 &c.= 3464701490190 4 6158398785417810 19
110296. 11157979270149012 04 17214582189 81
34650056
1

Consequently,

1116143772)1721458218979(1542326590,22

This third correction is carried two places beyond the extent of the divisor, for the sake of ascertaining rigidly the degree of accuracy now attained. For this purpose, we proceed thus: 628 &c. × 154 &c. =, 968, &c. is the true correction of the last divisor. Our anticipated correction was 1,000. For which if we substitute 968 &c. it will appear that our divisor should have ended in 1,678, &c. instead of 2. The error is, ,322 &c. which induces an ultimate error of (111 &c. : 154 &c. :: ,322 &c. &c. :),44 &c. Consequently, our third correction should be ..... 1542326590,66, &c. agreeing to 10 figures with the value previously determined. And the root is

$$x = 2.094551481542326590, \text{ &c.}$$

correct in the 18th decimal place at three approximations.

So rapid an advance is to be expected only under very favorable data. Yet this example clearly affixes to the new method, a character of unusual boldness and certainty; advantages derived from the overt manner of conducting the work, which thus contains its own proof.

The abbreviations used in the close of this example, are of a description sufficiently obvious and inartificial; but in order to perfect the algorithm of our method in its application to higher equations, and to the progress by simple digits, attention must be given to the following general principles of

Compendious Operation.

27. We have seen that every new digit of the root occasions the resolvend to be extended n figures to the right, and the m^{th} derivee $n-m$ figures; so that if the work be carried on as with a view to unlimited progress, every new

root-figure will be obtained and verified at an expence of $\frac{1}{2}(n.n+8)+2$ new lines of calculation, containing in all somewhat above $\frac{1}{2}(n.n-1.n+4)$ digits more than the preceding root-figure cost. But as the necessity for unlimited continuity can rarely, if ever, occur, we may consider ourselves at liberty to check the advance of the resolvend, as soon as it contains one or two figures more than the number we yet propose to annex to the root. This will happen, generally speaking, when $\frac{1}{n}$ th of the numeral places of the root are determined.

By arresting the advance of the resolvend, we diminish it in the first instance by an optional number (p) of places, and by n places more at each succeeding step. Neglecting at the same time an equal number of figures in the right hand places of each closing addend and its derivatives, as contributing nothing to the diminished resolvend, we thus cause the effective units' place of each derivee to retrograde* in the first instance $p+m-n$ places, and at every subsequent step, a number of places (m) equal to the index of the derivee.

In the mean time, while these amputations are diminishing the derivees and addends on the right hand, a uniform average diminution of one digit on the left hand is taking place in the successive classes of addends in each column. The obvious consequence is, that after about $\frac{1}{m+1}$ th of the root-

* As the advance of the closing addend is prepared by annexing dots to the superior preparatory addend, so its retrogradation may be prepared by a perpendicular line beginning before the proper place (the m^{th} or $m+p-n^{th}$) of the said preparatory addend, and continued indefinitely downward. One digit, or two, of those which fall to the right of this line in the next succeeding sum and preparatory addends, must be retained, for the sake ultimately of correctly adjusting the effective units of the subtrahend.

figures are found, the m^{th} derivee will receive no augmentation; or, in other words, it will be exterminated when $\frac{1}{m}$ th of those places are determined.

Again, when all the derivees inferior to M, the m^{th}, have vanished, the process reduces itself to that of the m^{th} order simply. For, the amount of the preparatory addends to L, the $m-1^{th}$ augmented derivee, will be $m-1$ times the previous closing addend ${}_{o}M$; and the preparatory addends to K, the $m-2^{th}$, will be formed from its previous closing addend ${}_{o}L$, by adding ${}_{o}Mr$ to it $m-2$ times successively; a procedure obviously similar to that with which the general synopsis commences.

28. From these principles we form the following conclusions, demonstrative of the facilities introduced by this improvement on the original process:

1. Whatever be the dimensions (n) of the proposed equation, whose root is to be determined to a certain number of places, only $\frac{1}{n}$ th part of that number (reckoning from the point at which the highest place of the closing addend begins to advance to the right of that of the first derivee) needs to be found by means of the process peculiar to the complete order of the equation; after which, $\frac{1}{n \cdot n-1}$ may be found by the process of the $n-1^{th}$ order, $\frac{1}{n-1 \cdot n-2}$ by that of the $n-2^{th}$ order, &c.

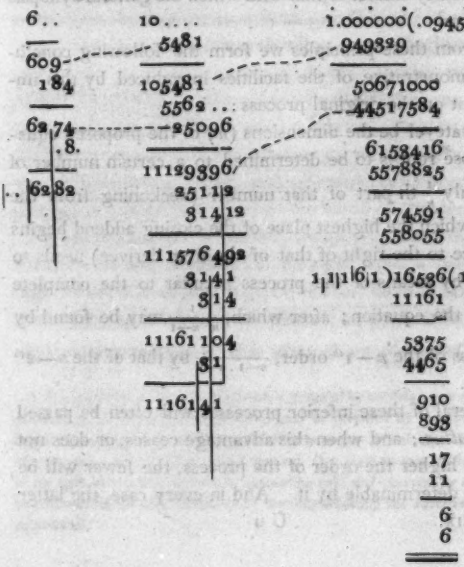
2. Several of these inferior processes will often be passed over *per saltum;* and when this advantage ceases, or does not occur, the higher the order of the process, the fewer will be the places determinable by it. And in every case, the latter

half of the root will be found by division simply. Meantime, the number of figures employed in verification of each successive root-digit, instead of increasing, is rapidly diminishing.

3. The process with which we commence, need not be of a higher order than is indicated by the number of places to which we would extend the root; and may be even reduced to an order as much lower as we please, by means of an introductory approximation.

Ex. III. Let the root of the equation in Ex. II. be determined to the tenth place of decimals.

Arranging the derivees as before, we proceed thus:



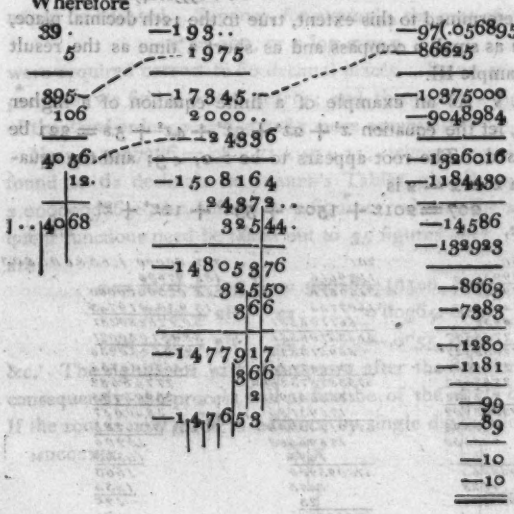
Consequently the root is 2.094551 4815, correct to the proposed extent, as appears on comparing it with the more enlarged value already found. The work occupied a very few minutes, and may be verified by mere perusal, as not a figure was written besides those which appear. By a similar operation, in less than half an hour, I have verified the root to the whole extent found in Ex. II.

Ex. IV. As a praxis in case of the intervention of negative numbers, let it be proposed to extract, to a convenient extent, that root of the equation $x^3 - 7x = -7$, whose first digits we have already determined to be 1.8. (See Art. 22.)

Making $x = 1.8 + z$, we have

$$-.097 = -1.98z + 3.9z^2 + z^3$$

Wherefore



Consequently the root is 1.85689867. This example was selected by LAGRANGE for its difficulty.

Of its three roots, that which we have now found is the most difficult to obtain; yet the whole work, including the preparatory portion in Art. 22, may be performed without one subsidiary figure, in less than a quarter of an hour.

A little attention and practice will render the mental aggregation of positive and negative numbers as familiar as the addition of either sort separately. The introduction of a small negative resolvend, instead of a large positive one, where the opportunity occurs, will then greatly abridge the operations. For example, the cube root of 2, or 1.26..

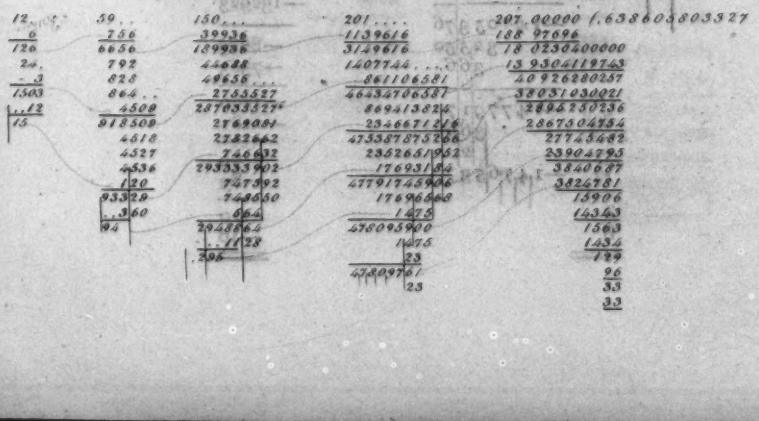
$$\begin{array}{r} -0078950105 \\ \hline 1.259921049895 \end{array}$$

was determined to this extent, true to the 19th decimal place, within as small a compass and as short a time as the result of Example III.

Ex. V. As an example of a finite equation of a higher order, let the equation $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 321$ be proposed. The root appears to be > 2, < 3; and the equation in $z = x - 2$ is

$$207 = 201z + 150z^2 + 59z^3 + 12z^4 + z^5$$

Hence,



Wherefore the root is 2.6386058033271, correct to the 19th decimal, and capable, like the former results, of being verified by simple inspection. The other roots are imaginary; for when $-x = .4$, the fourth derivee vanishes between two affirmative results, and when $-x = .7$ &c., the second disappears under similar circumstances. (Arts. 21, 22.)

It appears to me, that no explanation of this solution can be offered, which has not been abundantly anticipated in this Essay; and the student who peruses it in connection with the General Synopsis, and Arts. 23, 27, will acquire an indelible impression of the whole algorithm.

Ex. VI. If it were proposed to obtain a very accurate solution of an equation of very high dimensions, or of the irrational or transcendental kind, a plan similar to the following might be adopted. Suppose, for example, the root of

$$x^x = 100, \text{ or } x \log x = 2$$

were required correct to 60 decimal places. By an easy experiment we find $x = 3.6$ nearly; and thence, by a process of the *third* order, $x = 3.597285$ more accurately.

Now, $3597286 = 98 \times 71 \times 47 \times 11$, whose logarithms, found to 61 decimals in SHARPE's Tables, give R log R = 2.0000009663δ, &c. correct to 7 figures; whence the subsequent functions need be taken out to 55 figures only. They are

$a = \text{Mod} \div \log \text{R} = .9902694494\delta$08, &c.

$b = \text{Mod} \div 2\text{R} = \ldots\ldots\ldots 0'60364$, &c.

$c = -b \div 3\text{R} = \ldots\ldots\ldots -.0''55$, &c.

&c. The significant part disappears after the 8th derivee; consequently, the process will at first be of the *eighth* order. If the root is now made to advance by single digits, the first

of these will reduce the process to the *seventh* order; one more reduces it to the *sixth* order; two more, to the *fifth*, &c. The last 97 figures will be found by division alone.

But if the first additional correction is taken to 8 figures, and the second to 16, on the principle of Example II, we pass from the 8th order to the 4th at once, and thence to the 1st or mere division, which will give the remaining 89 figures. This mode appears in description to possess the greater simplicity, but is perhaps the more laborious.

It cannot fail to be observed, that in all these examples a great proportion of the whole labour of solution is expended on the comparatively small portion of the root, which is connected with the leading process. The toil attending this part of the solution, in examples similar in kind to the last, is very considerable; since every derivee is at this stage to receive its utmost digital extent. To obviate an unjust prejudice, I must therefore invite the reader's candid attention to the following particulars:

In all other methods the difficulty increases with the extent of the root, nearly through the whole work; in ours, it is in a great measure surmounted at the first step: in most others, there is a periodical recurrence to first conditions, under circumstances of accumulating inconvenience; in the new method, the given conditions affect the first derivees alone, and the remaining process is *arithmetically direct*, and increasingly easy to the end.

The question of practical facility may be decided by a very simple criterion; by comparing the *times* of calculation which I have specified, with a similar datum by Dr. HALLEY in favor of his own favorite method of approximation. (Philosophical Transactions for 1694.)

Addendum I. (*Vide* Art. 21.) *Note.* But in this case, it will be more elegant to find the differences at once by the theorem $\Delta^{t+1}D^{m}\varphi R' = \frac{m+1}{1}\Delta^{t}D^{m+1}\varphi R.r + \frac{m+1}{1}.\frac{m+2}{2}\Delta^{t}D^{m+2}\varphi R.r^{2} + \&c.$ which, supposing r to be constant, is a sufficiently obvious corollary to the theorem in Art. 7. All the results may then be derived from the first column by addition. Thus, for the latter transformations in Ex. II. Art. 22, the preparatory operation would be

1st. Terms.	Diff. 1st.	2nd.	3rd.
1000	—369	66	6
—400	63	6	
30	3		
1			

and the succeeding terms would be found by adding these differences in the usual way to the respective first terms.

Addendum II. It is with pleasure that I refer to the Imperial Encyclopædia (Art. Arithmetic) for an improved method of extracting the cube root, which should have been noticed in the proper place, had I been aware of its existence; but it was pointed out to me, for the first time, by the discoverer, Mr. EXLEY, of Bristol, after this Essay was completed. It agrees in substance with the method deduced in Art. 25, from my general principle, and affords an additional illustration of the affinity between that principle and the most improved processes of common arithmetic.

From the Press of
W. BULMER & Co.
Cleveland-row, St. James's,
London.